Lecture #1
Magnetic Catalysis: Basics

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Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports 576 (2015)
MAGNETIC CATALYSIS: PLAN OF LECTURES

- Dirac fermions in magnetic field
- Dimensional reduction
- Magnetic catalysis: basics
- Magnetic catalysis in toy model
- Magnetic catalysis in QED
- Magnetic catalysis in QCD
- Anisotropic confinement
- Inverse catalysis
- Phase diagram
QCD IN MAGNETIC FIELDS

• Relativistic collisions of *heavy ions* produce quark-gluon plasma & strong magnetic fields

\[ 10^{18} - 10^{19} \text{ Gauss} \ (\sqrt{|eB|} \sim 100 \text{ MeV}) \]

• Quark matter may form inside *magnetars*

\[ 10^{14} - 10^{16} \text{ Gauss} \ (\sqrt{|eB|} \sim 1 \text{ MeV to } 10 \text{ MeV}) \]

• Strong magnetic field is an instructive *theoretical tool* to study confined gauge theories such as QCD

\[ \geq 10^{19} \text{ Gauss} \ (\sqrt{|eB|} \geq 100 \text{ MeV to } 10 \text{ MeV}) \]
• Lagrangian density for charged Dirac fermions (units with $\hbar = 1$):

$\mathcal{L} = \bar{\psi} \left( i\gamma^\mu D_\mu - m \right) \psi$

where $D_\mu = \partial_\mu + i e A_\mu$, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$ and $g^{\mu\nu} = (1,-1,-1,-1)$

• Consider the following two types of global transformations:

$\psi \rightarrow e^{i\alpha} \psi$ and $\psi \rightarrow e^{i\alpha} \gamma^5 \psi$

where $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$

The corresponding Noether’s currents are

$j^\mu = \bar{\psi} \gamma^\mu \psi$ and $j_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$

They satisfy the relations:

$\partial_\mu j^\mu = 0$ and $\partial_\mu j_5^\mu = 2i m \bar{\psi} \gamma^5 \psi$

Both transformations are symmetries when $m = 0$, but chiral symmetry is broken when $m \neq 0$. [The chiral anomaly may complicate the situation]
**DIRAC VACUUM**

- $m = 0$: Dirac vacuum is a **semimetal**
  - No energy gap between the filled Dirac sea states and the empty positive-energy states ($E = \pm p$)
  - However, the density of states *vanishes* at $E=0$
  - A nonzero electric current could be produced by an arbitrarily small electric field

- $m \neq 0$: Dirac vacuum is an **insulator**
  - Energy gap $\Delta E = 2m$ between the antiparticle and particle states ($E = \pm \sqrt{p^2 + m^2}$)
  - the density of states @ $E=0$ *vanishes* (no states)
  - electric current is exponentially small, i.e.,
    $$e^{-\pi m^2/|eE|}$$ (due to Schwinger pair creation)
Dirac fermions at $B \neq 0$

- Dirac equation for charged fermions:
  \[
  (i\gamma^\mu D_\mu - m)\psi = 0
  \]
  where $A_\mu = \left(A_0, -\vec{A}\right)$ and the Landau gauge $\vec{A} = (-By, 0, 0)$ is used.
- Look for a solution in the form: $\psi = (i\gamma^\mu D_\mu + m)\phi$. Then,
  \[
  \left[-\partial_0^2 + (\partial_x + ieBy)^2 + \partial_y^2 + \partial_z^2 + i\gamma^1\gamma^2 eB - m^2\right]\phi = 0
  \]
- Normalized solutions for $\phi$ have the form
  \[
  \phi_{k,\pm} \propto 1 \pm isgn(eB)\gamma^1\gamma^2 \frac{1}{2} \varphi_k(y)e^{-i\omega t + ip_xx + ip_zz}
  \]
  where $\varphi_k$ are harmonic oscillator wave functions, i.e.,
  \[
  \varphi_k \propto H_k(\xi)e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l \text{ sgn}(eB) \quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}
  \]
- The dispersion relation is given by
  \[
  \omega = E_{n}^{\pm} = \pm \sqrt{2n|eB| + p_z^2 + m^2}
  \]
  where $n = k + \frac{1}{2} + \text{sgn}(eB)s_z$ and $s_z = \pm \frac{1}{2}$ is an eigenvalue of $\frac{i}{2}\gamma^1\gamma^2$
Degeneracy of Landau Levels

- The Landau level energies are independent of $p_x$
  \[ E_n^\pm = \pm \sqrt{2n|eB| + p_z^2 + m^2} \]
- This means that each level is highly degenerate
- Let’s calculate the degeneracy by confining the system in a finite box of size $L_x \times L_y$ with periodic boundary conditions
- The wave function is a plane wave in the $x$ direction: $\psi(x) \propto e^{ip_xx}$
  \[ \psi(0) = \psi(L_x) \implies e^{ip_xL_x} = 1 \implies p_x = \frac{2\pi n}{L_x}, \quad n = 1, 2, ..., N_{\text{max}} \]
- The value of $p_x$ sets the center of the Landau orbit in $y$-direction:
  \[ y_c \approx p_x l^2 \implies p_{x,\text{max}} l^2 \lesssim L_y \implies \frac{2\pi N_{\text{max}}}{L_x} \frac{1}{|eB|} \approx L_y \implies \frac{N_{\text{max}}}{L_x L_y} \approx \frac{|eB|}{2\pi} \]
- The degeneracy is proportional to the field strength and the size (area) of the system in the spatial directions perpendicular to $\vec{B}$
  \[ N_{\text{max}} \approx \frac{|eB|}{2\pi} L_x L_y \]
Landau energy levels at $m = 0$

$$E_n^\pm = \pm \sqrt{2n|eB| + \frac{p_z^2}{\hbar^2}}$$

where $n = k + \frac{1}{2} + \text{sgn}(eB)s_z$

- Orbital \quad Spin

- Lowest Landau level is spin polarized

$$E_0^\pm = \pm p_z \quad (k = 0, \ s_z = -\frac{1}{2})$$

- Density of states at $E=0$:

$$\left. \frac{dn}{dE} \right|_{E=0} = \frac{|eB|}{2\pi} \frac{1}{2\pi} = \frac{|eB|}{4\pi^2}$$

- Higher Landau levels ($n \geq 1$) are twice as degenerate:

  (i) $k = n \quad \& \quad s = -\frac{1}{2}$

  (ii) $k = n - 1 \quad \& \quad s = +\frac{1}{2}$
By definition,
\[ G(r, r') = i \langle r \mid (i\gamma^\mu D_\mu - m)^{-1} \mid r' \rangle \]
\[ = i(i\gamma^\mu D_\mu + m)_r \langle r \mid [(i\gamma^\mu D_\mu - m)(i\gamma^\nu D_\nu + m)]^{-1} \mid r' \rangle \]
\[ = i(i\gamma^\mu D_\mu + m)_r \langle r \mid [-D^\mu D_\mu + i\gamma^1 \gamma^2 eB - m^2]^{-1} \mid r' \rangle \]
\[ = i(i\gamma^\mu D_\mu + m)_r \sum \langle r \mid k, p_z, s_z \rangle (\omega^2 - E_n^2)^{-1} \langle k, p_z, s_z \mid r' \rangle \]

Note that the explicit form of the wave functions is the same as before
\[ \psi_{k, p_z, s_z}(r) = \langle r \mid k, p_z, s_z \rangle \propto H_k(\xi) e^{-\frac{\xi^2}{2}} e^{-i\omega t + ip_z z} U_{s_z}, \text{ where } \xi = \frac{y}{l} + p_x l \]

The final expression for the propagator has the form
\[ G(\omega, p_z; \vec{r}_\perp, \vec{r}'_\perp) = e^{i\Phi(\vec{r}_\perp, \vec{r}'_\perp)} \tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}'_\perp) \]
where \( \Phi(\vec{r}_\perp, \vec{r}'_\perp) = -e \int^{\vec{r}'_\perp \rightarrow \vec{r}_\perp}_{\vec{r}_\perp} A_\nu dr^\nu \) is the Schwinger phase (!), and
\[ \tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}'_\perp) = \int \frac{d^2 \vec{p}_\perp}{(2\pi)^2} e^{ip_\perp \cdot (\vec{r}_\perp - \vec{r}'_\perp)} \tilde{G}(\omega, \vec{p}) \]
The Fourier transform of the translation invariant part reads
\[ \tilde{G}(\omega, \vec{p}) = ie^{-\vec{p}_\perp^2 l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega, \vec{p})}{\omega^2 - E_n^2} \]

where
\[ D_n(\omega, \vec{p}) = 2(\omega \gamma^0 - p_z \gamma^3 + m) [\mathcal{P}_+ L_n(2\vec{p}_\perp^2 l^2) - \mathcal{P}_- L_{n-1}(2\vec{p}_\perp^2 l^2)] \]
\[ + 4(\vec{p}_\perp \cdot \hat{\gamma}_\perp)L_{n-1}^1(2\vec{p}_\perp^2 l^2) \]
and the following notation for the spin projectors is used
\[ \mathcal{P}_\pm = \frac{1 \pm isgn(eB)\gamma^1\gamma^2}{2} \]

Similarly, in momentum-coordinate space representation:
\[ \tilde{G}(\omega, p_z; \vec{r}_\perp) = i \frac{e^{-\vec{r}_\perp^2/(4l^2)}}{2\pi l^2} \sum_{n=0}^{\infty} \frac{F_n(\omega, p_z; \vec{r}_\perp)}{\omega^2 - E_n^2} \]

where
\[ F_n(\omega, p_z; \vec{r}_\perp) = 2(\omega \gamma^0 - p_z \gamma^3 + m) \left[ \mathcal{P}_+ L_n\left(\frac{\vec{r}_\perp^2}{2l^2}\right) - \mathcal{P}_- L_{n-1}\left(\frac{\vec{r}_\perp^2}{2l^2}\right) \right] \]
\[ - \frac{i}{l^2} (\vec{r}_\perp \cdot \hat{\gamma}_\perp)L_{n-1}^1\left(\frac{\vec{r}_\perp^2}{2l^2}\right) \]
**Dimensional Reduction**

- The low-energy dynamics is determined by the lowest Landau level \((n=0)\)
  \[
  E_0^\pm = \pm p_z
  \]
- This is a \((1+1)D\) spectrum!
- Propagator is also \((1+1)D:\)
  \[
  \tilde{G}_{LLL}(\omega, \mathbf{p}) = 2ie^{-\mathbf{p}_\perp^2 \ell^2} \frac{\omega \gamma^0 - p_z \gamma^3}{\omega^2 - p_z^2} \frac{1 + isgn(eB)\gamma^1 \gamma^2}{2}
  \]
- In addition, there is a nonzero density of states at \(E=0:\)
  \[
  \left. \frac{dn}{dE} \right|_{E=0} = \frac{1}{\delta E} \left( \frac{N_{\text{max}}}{L_x L_y} \right) \left( \int_0^{\delta E} \frac{dp_z}{2\pi} \right) = \frac{|eB|}{4\pi^2}
  \]
**PAIRING INSTABILITY**

- Thought experiment:
  - Create a particle-antiparticle pair (energy price: $\Delta E$)
  - The pair can form a bosonic bound state (energy gain: $-\epsilon_b$)
  - If $\epsilon_b > \Delta E$, copious formation of bound states is beneficial
  - Note, $\Delta E$ can be arbitrarily small when $m = 0$ (!)
  - The bound states of fermions are bosons
  - Bosons can (and will) occupy the lowest energy state ($\vec{P} = 0$), and thus form a Bose condensate $\langle \bar{\psi}\psi \rangle \neq 0$
  - Ground state (vacuum) changes its properties (e.g., chiral symmetry breaks down, an energy gap opens in spectrum)
Do bound states always form in 3D?

- Consider a 3D potential well in quantum mechanics
  \[ U(r) = \begin{cases} -g \frac{\pi^2 \hbar^2}{8m^*a^2} & \text{for } r \leq a \\ 0 & \text{for } r > a \end{cases} \]

- Bound states form only when the well is deep enough (namely, \( g > 1 \)): \[ |E_{3D}| \approx \frac{\pi^4 \hbar^2}{2^7 a^2 m^*} (g - 1)^2, \text{ assuming } 0 < g - 1 << 1 \]

- There are no bound states when \( g < 1 \), i.e., when the well is not deep enough (in other words, when the coupling constant is not strong enough)
**COMPARE: BOUND STATES IN 1D**

- Bound states always form

\[ |E_{1D}| \approx \frac{m_*}{2\hbar^2} \left( -\int_{-\infty}^{+\infty} U(x) \, dx \right)^2 \]

- This is a perturbative result (!)

\[ |E_{1D}| \propto g^2 , \text{ when } U(x) \rightarrow gU(x) \]

- Rigorous statement: at least one bound state exists if

\[ \int (1 + |x|) |U(x)| \, dx < \infty \quad \& \quad \int U(x) \, dx \leq 0 \]

[B. Simon, Annals Phys. 97 (1976) 279]
**How about bound states in 2D?**

- Bound states always form

\[
|E_{2D}| \approx \frac{\hbar^2}{a^2 m_*} \exp \left( - \frac{\hbar^2}{m_*} \int_0^\infty r U(r) dr \right)^{-1}
\]

- This is a non-perturbative result

\[
|E_{2D}| \propto \exp \left( - \frac{C}{g} \right), \text{ when } U(x) \to gU(x)
\]

- Rigorous statement: at least one bound state exists if

\[
\int |U(x)|^{1+\varepsilon} d^2x < \infty, \quad \int (1 + x^2)^{\varepsilon}|U(x)| d^2x < \infty \quad \& \quad \int U(x) \, d^2x \leq 0
\]

[B. Simon, Annals Phys. 97 (1976) 279]
**Universal Magnetic Catalysis**

- Quantum field theory of charged fermions \((m=0)\) at \(B \neq 0\)
  - Dimensional reduction (caused by a nonzero \(B\))
  - Nonzero density of states \(\propto |eB|\) at \(E=0\)
  - Attraction between particles and antiparticles
- Universal outcome:
  - Copious particle-antiparticle pairing at low energies
  - Condensation of boson pairs that destabilizes the trivial Dirac vacuum
  - Spontaneous rearrangement of the ground state
  - Breakdown of chiral symmetry
  - Opening a nonzero gap in the Dirac spectrum

\[\text{[Gusynin, Miransky, Shovkovy, Phys. Rev. Lett. 73, 3499 (1994)]}\]

- The mechanism is similar to superconductivity in metals due to Cooper pairing of electrons
Lecture #2
Magnetic Catalysis in a Toy Model

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Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports 576 (2015)
Let us consider a Nambu-Jona-Lasino model \((m = 0)\) with four-fermion contact interaction

\[
\mathcal{L} = \bar{\psi} \left( i\gamma^\mu D_\mu \right) \psi + \frac{G}{2} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2 \right]
\]

After the Hubbard–Stratonovich transformation, this is equivalent to

\[
\mathcal{L} = \bar{\psi} \left( i\gamma^\mu D_\mu - \sigma - i\gamma^5\pi \right) \psi - \frac{\sigma^2 + \pi^2}{2G}
\]

where the following composite fields were introduced

\[\sigma = -G \bar{\psi}\psi \quad \text{and} \quad \pi = -G \bar{\psi}i\gamma^5\psi\]

The effective action for the composite fields reads

\[
\Gamma(\sigma, \pi) = -\frac{1}{2G} \int d^4x (\sigma^2 + \pi^2) - i \text{Tr} \ln \left[ i\gamma^\mu D_\mu - \sigma - i\gamma^5\pi \right]
\]
SYMMETRY OF THE MODEL

• \( U_L(1) \) symmetry transformations, \( \psi \to e^{i\alpha_L (1-\gamma^5)/2} \psi \)
  \[ \bar{\psi}\psi \to \cos \alpha_L \bar{\psi}\psi - \sin \alpha_L \bar{\psi}i\gamma^5\psi \]
  \[ \bar{\psi}i\gamma^5\psi \to \sin \alpha_L \bar{\psi}\psi + \cos \alpha_L \bar{\psi}i\gamma^5\psi \]

• \( U_R(1) \) symmetry transformations, \( \psi \to e^{i\alpha_R (1+\gamma^5)/2} \psi \)
  \[ \bar{\psi}\psi \to \cos \alpha_R \bar{\psi}\psi + \sin \alpha_R \bar{\psi}i\gamma^5\psi \]
  \[ \bar{\psi}i\gamma^5\psi \to -\sin \alpha_R \bar{\psi}\psi + \cos \alpha_R \bar{\psi}i\gamma^5\psi \]

• In terms of the composite fields, \( U_L(1) / U_R(1) \) transformations:
  \[ \sigma \to \cos \alpha_L \sigma - \sin \alpha_L \pi \]
  \[ \pi \to \sin \alpha_L \pi + \cos \alpha_L \sigma \]

(Note that \( \rho^2 = \sigma^2 + \pi^2 \) remains an invariant.)

• Just like the original action \( \int \mathcal{L} \, d^4x \), the effective action \( \Gamma(\sigma, \pi) \) should be invariant under the symmetry transformations, i.e.,
  \[ \Gamma(\sigma, \pi) = \Gamma(\rho) + \frac{1}{2} f_1^{\mu\nu} (\partial_\mu \sigma \partial_\nu \sigma + \partial_\mu \pi \partial_\nu \pi) + \cdots \]
Let us consider a homogeneous ground state with a uniform $\sigma$

$$\sigma = -G \langle \bar{\psi} \psi \rangle \neq 0$$

(Because of the chiral symmetry, we can always set $\pi = 0$.)

In this case, $\Gamma(\sigma) = -\int V(\sigma) d^4x$, where the effective action is

$$V(\sigma) = \frac{\sigma^2}{2G} - \frac{i}{2} \int_0^\infty \frac{ds}{s} \text{tr} \left( x \left| e^{-is(D^\mu D_\mu - i\gamma^1 \gamma^2 eB + \sigma^2)} \right| x \right) - (\infty)$$

By using the Schwinger result [Phys. Rev. 82, 664 (1951)]

$$\left\langle x \left| e^{-is(D^\mu D_\mu - i\gamma^1 \gamma^2 eB + \sigma^2)} \right| x \right\rangle = \frac{e^{-is\sigma^2 - i\pi/4}}{8(\pi s)^{3/2}} eBs [\cot eBs + \gamma^1 \gamma^2]$$

We derive the effective potential (after $s \to -is$):

$$V(\sigma) = \frac{\sigma^2}{2G} + \frac{eB}{8\pi^2} \int_1^{\Lambda^2} \frac{ds}{s^2} e^{-s\sigma^2} \coth eBs - (\infty)$$
Effective Potential: Results

Lowest energy ground state is defined by: \( \frac{dV(\sigma)}{d\sigma} = 0 \) (gap equation)

At weak coupling \( (G \to 0) \), the analytical solution for the minimum

\[
\sigma_{\text{min}} \approx \frac{eB}{\pi} \exp\left(\frac{\Lambda^2}{|eB|}\right) \exp\left(-\frac{4\pi^2}{|eB|G}\right)
\]
In fact, the gap equation at $B=0$ reads

$$\frac{G\Lambda^2 - 4\pi^2}{G} = \sigma^2 \ln \frac{\Lambda^2}{\sigma^2}$$

It has a nontrivial solution $\sigma_{\min} \neq 0$ only when the coupling strength is sufficiently strong, i.e., $G > G_c = 4\pi^2/\Lambda^2$
Dynamical mass

- Recall: \[ \mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - \sigma - i \gamma^5 \pi \right) \psi - \frac{\sigma^2 + \pi^2}{2G} \]
- The ground state expectation value \( \langle \sigma \rangle = \sigma_{\text{min}} \) determines the dynamical mass of fermions \( m_{\text{dyn}} \) in the new Dirac vacuum.

- Also, the chiral symmetry is broken in a state with \( \langle \sigma \rangle \neq 0 \)
Nambu-Goldstone Bosons

- When a continuous global symmetry breaks down, massless Nambu-Goldstone bosons appear in the particle spectrum

\[(D_\pi)^{-1} = \frac{\delta^4(x)}{G} + i \text{tr}[G(x,0)i\gamma^5 G(0,x)i\gamma^5]\]

- The dispersion relation of NG bosons at \(\vec{p} \rightarrow 0\)

\[E_\pi = \sqrt{\nu_{\pi,\perp}^2 \vec{p}_{\perp}^2 + p_Z^2}\]

where \(\nu_{\pi,\perp} \ll 1\) at weak coupling

- The relation for the \(\sigma\)-boson

\[E_\sigma = \sqrt{M_\sigma^2 + \nu_{\sigma,\perp}^2 \vec{p}_{\perp}^2 + p_Z^2}\]

where \(M_\sigma = 2\sqrt{3}m_{dyne} \quad \text{and} \quad \nu_{\pi,\perp} \ll 1\)
**NONZERO TEMPERATURE**

- Partition function:

\[
Z_{T,\mu} = \text{Tr} \left[ \exp \left( - \frac{H - \mu N}{T} \right) \right]
\]

\[
= \int [d\psi d\bar{\psi} d\sigma d\pi] \exp \left( \int_0^{-i/T} dt \int d^3x \left[ \bar{\psi} (i\gamma^\mu D_\mu - \sigma - i\gamma^5\pi)\psi - \frac{\sigma^2 + \pi^2}{2G} \right] \right)
\]

where the fermion/boson fields satisfy (anti)periodic boundary conditions in imaginary time, e.g., \(\psi(0) = -\psi(-i/T)\)

- **Note #1:** \(Z_{T,\mu}\) is similar to the generating functional at \(T=0\)

- **Note #2:** Hubbard–Stratonovich trick \(\iff\) Gaussian integral

- The effective potential is similar to that at \(T=0\), but with the energy integration replaced by the Matsubara sum:

\[
\int_0^\infty \frac{d\omega}{2\pi} e^{is\omega^2} \to iT \sum_{n=-\infty}^{\infty} e^{is(i\omega_n)^2} \]

where \(\omega \to i\omega_n = i\pi T(2n + 1)\)
EFFECTS OF NONZERO TEMPERATURE

\[ V_{\beta, \mu}(\rho) = V(\rho) - \frac{N}{2\beta \pi^2 l^2} \int_0^\infty dk_3 \left\{ \ln \left[ 1 + e^{-\beta \left( \sqrt{\rho^2 + k_3^2} - \mu \right)} \right] + 2 \sum_{n=1}^\infty \ln \left[ 1 + e^{-\beta \left( \sqrt{\rho^2 + k_3^2 + 2n/l^2} - \mu \right)} \right] + (\mu \to -\mu) \right\} \]
Notice that at $T = 0$ the chemical potential $\mu$ has no effect on the effective potential when $\sigma > \mu$ (This is not true at $T \neq 0$)
• Effective potential for the composite field, e.g., $\sigma = -G \bar{\psi} \psi$

$$\frac{dV(\sigma)}{d\sigma} = 0 \quad \text{(gap equation)}$$

• In NJL, e.g., $V_{NJL}(\sigma) = \frac{\sigma^2}{2G} + i \text{ tr ln}[i\gamma^\mu D_\mu - \sigma]$, giving

$$\frac{\sigma}{G} - i \text{ tr} \left[\frac{1}{i\gamma^\mu D_\mu - \sigma}\right] = 0 \quad \Rightarrow \quad \sigma = G \text{ tr}[G(x, x)]$$

• The same gap equation can be obtained from the Schwinger-Dyson equation for the fermion self-energy/propagator

$$G^{-1}(x, x') - G_0^{-1}(x, x') = -iG \sum_i \Gamma_i [G(x, x)\Gamma_i - \text{tr}{G(x, x)\Gamma_i}] \Gamma_i \delta^4(x - x')$$

where ansatz $G^{-1}(x, x') = -i \left(i\gamma^\mu D_\mu - m_{dyn}\right)\delta^4(x - x')$ is used
**Another way: pion as a bound state**

- Homogeneous Bethe-Salpeter equation for a *massless* bound state with quantum numbers of the NG boson

\[
\chi^\beta = \chi^\beta + \chi^\beta
\]

- As we’ll see, in NJL model in the strong-field limit, the pion’s wave function in momentum space should have the structure:

\[
\chi(p; P \to 0) = A(p_\parallel)e^{-p_\perp^2 t^2} \frac{\omega \gamma^0 - p_z \gamma^3 - m}{\omega^2 - p_Z^2 - m^2} \gamma^5 \gamma^0 - p_z \gamma^3 - m \frac{\omega^2 - p_Z^2 - m^2}{\omega^2 - p_Z^2 - m^2}
\]

where \( A(p_\parallel) \) with \( p_\parallel = (\omega, p_z) \) satisfies a simple integral equation

\[
A(p_\parallel, E) = \frac{G |eB|}{4\pi^3} \int \frac{A(k_\parallel, E) d^2 k_\parallel, E}{k_\parallel, E^2 + m^2}
\]

(here mass parameter \( m \) is treated as a variational parameter)
It is instructive to recast the problem in terms of

\[ \Psi(r_\parallel) = \int \frac{d^2k_\parallel}{(2\pi)^2} \frac{e^{-ir_\parallel \cdot k_\parallel}}{k_\parallel^2 + m^2} A(k_\parallel) \]

Function \( \Psi(r_\parallel) \) satisfies the following 2D Schrodinger equation:

\[ \left[ -\nabla_{r_\parallel}^2 + m^2 + V(r_\parallel) \right] \Psi(r_\parallel) = 0 \]

where \(-m^2\) plays the role of energy \( \epsilon \), and \( V(r_\parallel) \) is a model-dependent potential (as we will see later).

In the NJL model, \( V(r_\parallel) \) is proportional to a \( \delta \)-function

\[ V(r_\parallel) = -\frac{G|eB|}{\pi} \delta_\Lambda^2(r_\parallel) = -\frac{G|eB|}{\pi} \int_0^\Lambda \frac{d^2k_\parallel}{(2\pi)^2} e^{-ir_\parallel \cdot k_\parallel} \]

There exists a bound state solution (\( \epsilon_b < 0 \)) in this Schrodinger problem and, thus, also a real solution for \( m \), i.e.,

\[ m^2 = -\epsilon_b \approx \Lambda^2 \exp \left( -\frac{4\pi^2}{|eB|G} \right) \]  

(LLL & weak coupling)
Lecture #3
Magnetic Catalysis in QED

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Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports 576 (2015)

Summer School on Frontiers in Theoretical Physics and Huada School on QCD:
Chiral Anomaly and Strong Magnetic Fields in Heavy-ion Collisions
**MAGNETIC CATALYSIS IN QED**

- Lagrangian density invariant under $\text{SU}_L(N_f) \times \text{SU}_R(N_f) \times \text{U}(1)$

$$
\mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} + \bar{\psi}_f (i\gamma^\mu D_\mu) \psi_f
$$

where $D_\mu = \partial_\mu + ie(A_\mu + a_\mu)$ and $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

- The Bethe–Salpeter equation for NG states ($\beta = 1, \ldots, N_f^2 - 1$):

$$
\chi^\beta_{AB}(u, u'; P) = -i \int d^4 u_1 d^4 u'_1 d^4 u_2 d^4 u'_2 G_{AA_1}(u, u_1) K_{A_1 B_1; A_2 B_2}(u_1 u'_1, u_2 u'_2) \chi^\beta_{A_2 B_2}(u_2, u'_2; P) G_{B_1 B}(u'_2, u')
$$

where the wave function is defined by $\chi^\beta_{AB} = \langle 0 | T \psi_A(u) \bar{\psi}_B(u') | P; \beta \rangle$

Diagrammatically

where the kernel (in the ladder approximation) is

$$
K_{A_1 B_1; A_2 B_2}(u_1 u'_1, u_2, u'_2) = -4\pi i\alpha \delta_{a_1 a_2} \delta_{b_2 b_1} \gamma^\mu_{n_1 n_2} \gamma^\nu_{m_2 m_1} D_{\mu\nu}(u'_2 - u_2) \delta(u_1 - u_2) \delta(u'_1 - u'_2)
$$

$$
+ 4\pi i\alpha \delta_{a_1 b_1} \delta_{b_2 a_2} \gamma^\mu_{n_1 m_1} \gamma^\nu_{m_2 n_2} D_{\mu\nu}(u_1 - u_2) \delta(u_1 - u'_1) \delta(u_2 - u'_2)
$$

Hartree term plays no role for NG bound states
Solution in Strong Field Limit

- Structure of the NG-boson wave function \((r_\mu = u_\mu - u'_\mu)\):
  \[ \chi^\beta_{AB}(u, u'; P) = \lambda^\beta_{ab} e^{-iPR} \exp \left[ -i \epsilon^{\mu} A^{\text{ext}}_\mu (R) \right] \tilde{\chi}_{nm}(R, r; P) \]

- In the LLL approximation, the equation reduces to
  \[
  \varphi(p_{||}) = \frac{\pi \alpha}{(2\pi)^4} \int d^2 k_{||} (1 - i \gamma^1 \gamma^2) \gamma^\mu \frac{k_{||}}{k_{||}^2 - m_{\text{dyn}}^2} \varphi(k_{||}) \gamma^\nu \left(1 - i \gamma^1 \gamma^2\right) D_{\mu\nu}^{\parallel}(k_{||} - p_{||})
  \]
  where we introduced \((\hat{p}_{||} - m_{\text{dyn}}) \tilde{\chi}(p)(\hat{p}_{||} - m_{\text{dyn}}) = \exp(-l^2 p_{\perp}^2) \varphi(p_{||})\)

- The solution should have the following Dirac structure
  \[ \varphi(p_{||}) = A \gamma_5 (1 - i \gamma^1 \gamma^2) \]

- Finally, the equation for \(A(p_{||})\) reads
  \[
  A(p_{||}) = \frac{\alpha}{2\pi^2} \int \frac{A(k_{||}) d^2 k_{||}}{k_{||}^2 + m_{\text{dyn}}^2} \int_0^\infty dx \frac{e^{-xl^2/2}}{x + (k_{||} - p_{||})^2}
  \]

Compare with the NJL model
Rewrite the problem in terms of
\[ \Psi(r) = \int \frac{d^2k_\parallel}{(2\pi)^2} \frac{e^{ir \cdot k_\parallel}}{k_\parallel^2 + m_{dyn}^2} A(k_\parallel) \]

Function \( \Psi(r) \) satisfies the following 2D Schrodinger equation:
\[ \left[ -\nabla_r^2 + m_{dyn}^2 + V(r) \right] \Psi(r) = 0 \]

where
\[ V(r) = -\frac{\alpha}{2\pi^2} \int d^2p e^{ip \cdot r} \int_0^\infty dx \frac{\exp(-x/2)}{l^2 p^2 + x} = \frac{\alpha}{\pi l^2} \exp\left(\frac{r^2}{2l^2}\right) \text{Ei}\left(-\frac{r^2}{2l^2}\right) \]

The potential is long-ranged with the following asymptote
\[ V(r) \simeq -\frac{2\alpha}{\pi} \frac{1}{r^2}, \quad r \to \infty \]

The lowest energy bound state gives
\[ m_{dyn} \simeq C \sqrt{|eB|} \exp\left[-\frac{\pi}{2} \left(\frac{\pi}{2\alpha}\right)^{1/2}\right] \quad (\text{LLL & weak coupling}) \]
Photon exchange interaction is screened in a strong B-field

$$\mathcal{D}_{\mu\nu}^{-1}(u, u') = D_{\mu\nu}^{-1}(u - u') + \Pi_{\mu\nu}(u, u')$$

where \( \Pi_{\mu\nu} \equiv (q_{\mu}q_{\nu} - q_{\mu \parallel}g_{\mu\nu}) \approx e^{-q_{\perp}^2l^2}\Pi(q_{\parallel}^2) \)

Then, the screened photon propagator reads

$$\mathcal{D}_{\mu\nu}(q) = -i \left[ \frac{1}{q^2 g_{\mu\nu}} + \frac{q_{\mu \parallel}q_{\nu \parallel}}{q^2 q_{\parallel}^2} + \frac{1}{q^2 + q_{\parallel}^2 \Pi(q_{\perp}^2, q_{\parallel}^2)} \left( g_{\mu\nu} - \frac{q_{\mu \parallel}q_{\nu \parallel}}{q_{\parallel}^2} \right) - \frac{\lambda}{q^2} \frac{q_{\mu}q_{\nu}}{q_{\parallel}^2} \right]$$

where the polarization function has the asymptotes

$$\Pi(q_{\parallel}^2) \simeq \frac{\bar{\alpha}}{3\pi} \frac{|eB|}{m_{\text{dyn}}^2}, \quad \text{as } |q_{\parallel}^2| \ll m_{\text{dyn}}^2 \quad \text{(extremely narrow range in } q_{\parallel}^2)$$

$$\Pi(q_{\parallel}^2) \simeq -\frac{2\bar{\alpha}}{\pi} \frac{|eB|}{q_{\parallel}^2}, \quad \text{as } |q_{\parallel}^2| \gg m_{\text{dyn}}^2 \quad \Rightarrow \quad \frac{1}{q^2 + q_{\parallel}^2 \Pi(q_{\perp}^2, q_{\parallel}^2)} \simeq 1 \frac{1}{q^2 - M_{\gamma}^2}$$

where the effective photon screening mass is

$$M_{\gamma}^2 = \frac{2\bar{\alpha}}{\pi} |eB|$$
Let us re-analyze the problem with screening

\[ A(p_{\parallel}) = \frac{\alpha}{4\pi^2} \int \frac{A(k_{\parallel}) d^2 k_{\parallel}}{k_{\parallel}^2 + m^2} \int_0^\infty dx \left( \frac{e^{-x l^2/2}}{x + (k_{\parallel} - p_{\parallel})^2} + \frac{e^{-x l^2/2}}{x + (k_{\parallel} - p_{\parallel})^2 + M_Y^2} \right) \]

• Improved vs. simple ladder approximations: \( \alpha \rightarrow \alpha/2 \)

• Note, the dynamical mass is very sensitive to small \( \alpha \) (or \( \alpha/2 \)):

\[ m_{\text{dyn}} \simeq C \sqrt{|eB|} \exp \left[ -\frac{\pi}{2} \left( \frac{\pi}{2\alpha} \right)^{1/2} \right] \quad \text{(ladder approximation)} \]

and, thus, changes drastically with inclusion of screening

• The bigger problem is that the improved ladder approximation is not reliable either
  
  – The vertex corrections will change the result too
  
  – Singularities \( \sim \ln (|eB|/m_{\text{dyn}}^2) \sim 1/\sqrt{\alpha} \) in higher-order diagrams

• Re-summation of infinitely many diagrams is needed (!)
QED in a strong field looks almost like (1+1)D

Lesson from exactly solvable (1+1)D Schwinger model: find a gauge in which all (singular) vertex corrections vanish!

Such a (non-local) gauge exists

\[ D_{\mu\nu}(q) = -i \frac{1}{q^2} \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) - i d(q^2, q^2) \frac{q^\parallel q^\parallel}{q^2 q^2} \]

where

\[ d = -q^2 \Pi / [q^2 + q^\parallel \Pi] + q^2 / q^2 \]

The corresponding full photon propagator reads

\[ D_{\mu\nu}(q) = -i \frac{g_{\mu\nu}}{q^2 + q^\parallel \Pi (q^\perp, q^\parallel)} - i \frac{g_{\mu\nu}^\perp}{q^2} + i \frac{q^\perp q^\perp + q^\parallel q^\parallel + q^\parallel q^\perp}{(q^2)^2} \]

All potentially dangerous infrared singularities vanish because

\[ P_+ \gamma_\mu P_+ = \gamma_{\parallel, \mu} \quad \text{and} \quad \gamma_{\parallel, \alpha} \gamma_{\parallel, \mu_1} \gamma_{\parallel, \mu_2} \ldots \gamma_{\parallel, \mu_{2n+1}} \gamma_{\parallel} = 0 \]
Let us use the method of Schwinger-Dyson equation this time:

\[ \tilde{G}(x) = \tilde{G}_0(x) - 4\pi\alpha \int d^4y d^4z \ e^{-i\Phi(x,y) - i\Phi(y,z)} \tilde{G}_0(x-y)\gamma^\mu \tilde{G}(y-z)\gamma^\nu \tilde{G}(z)D_{\mu\nu}(y-z) \]

where all Schwinger phases were carefully accounted for, and the nonlocal gauge is assumed in the photon propagator

\[ D_{\mu\nu}^{-1}(x - y) = D_{\mu\nu}^{-1}(x - y) - 4\pi\alpha \text{tr} [\gamma_\mu \tilde{G}(x - y)\gamma_\nu \tilde{G}(y - x)] \]

Perform Fourier transform and use LLL approximation,

\[ \tilde{G}_0(p_{||}) = 2ie^{-\vec{p}_{\perp}^2l^2} \frac{\hat{p}_{||}}{p_{||}^2} P_+ \quad \text{and} \quad \tilde{G}(p_{||}) = 2ie^{-\vec{p}_{\perp}^2l^2} \frac{\hat{p}_{||} + A(p_{||})}{p_{||}^2 - A^2(p_{||})} P_+ \]

Derive the following gap equation:

\[ A(p_{||}) = \frac{\alpha}{2\pi^2} \int \frac{d^2 k_{||} A(k_{||})}{k_{||}^2 + A^2(p_{||})} \int_0^\infty dx \frac{e^{-xl^2/2}}{x + (k_{||} - p_{||})^2 + M_Y^2 e^{-xl^2/2}} \]

Compare with the gap equations in the (improved) ladder QED, obtained with Bethe-Salpeter method.
• The numerical result is fitted well by

\[ m_{\text{dyn}} \approx \sqrt{2|eB|} (\alpha N_f)^{1/3} \exp \left[-\frac{\pi}{\alpha \ln \frac{C_1}{\alpha N_f}}\right], \quad C_1 \approx 1.82 \pm 0.06 \]
QCD IN MAGNETIC FIELD

• QCD is strongly coupled & nonperturbative

• There are theoretical tools that provide insight
  – High-energy (weak-coupling) expansion
  – Large $N_c$ expansion
  – High temperature limit ($T \gg \Lambda_{\text{QCD}}$)
  – High density limit ($\mu \gg \Lambda_{\text{QCD}}$)
  – Lattice QCD

• Strong magnetic field $B$ is yet another tool
  – it probes physics at short distances $\ell \sim 1/\sqrt{|eB|}$
• Lagrangian density of QCD in an external magnetic field

\[ \mathcal{L} = -\frac{1}{2} F_{\mu\nu}^A F_{\mu\nu}^A + \bar{\psi}_f (i \gamma^\mu D_\mu) \psi_f \]

where \( D_\mu = \partial_\mu + igA_\mu^A \lambda^A / 2 + ie_f A_{\mu}^{\text{ext}} \)

\[ F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + gf^{ABC} A_\mu^B A_\nu^C \]

• The global chiral symmetry of the model

\[ SU_L(N_u) \times SU_R(N_u) \times SU_L(N_d) \times SU_R(N_d) \times U_A(-) (1) \]

- chiral symmetry of up-flavors
- anomaly-free combination of \( U_A^{(u)} (1) \) and \( U_A^{(d)} (1) \)
- chiral symmetry of down-flavors

• Quark masses \( m_u \neq m_d \neq 0 \) break the symmetry down to

\[ SU_V(N_u) \times SU_V(N_d) \]
**RUNNING COUPLING & CONFINEMENT**

- Coupling constant in QCD runs with the energy scale,

\[
\frac{1}{\alpha_s(\mu)} \simeq b \ln \frac{\mu^2}{\Lambda_{QCD}^2}, \quad \text{where} \quad b = \frac{11 N_c - 2 N_f}{12\pi}
\]

- The question is: What happens in a strong magnetic field?
QCD in strong B-field

- Energy scales in the problem at hand

confined gluodynamics, glueballs

Magnetic catalysis in weakly coupled QCD and strong B-field, strong gluon screening

pure (anisotropic) gluodynamics, all massive quarks decoupled,

\[
\frac{1}{\alpha_s(\mu)} \approx b_0 \ln \frac{\mu^2}{\Lambda_{QCD}^2}
\]

deep-UV region with asymptotic freedom and weak B-field

\[
\frac{1}{\alpha_s(\mu)} \approx b \ln \frac{\mu^2}{\Lambda_{QCD}^2}
\]
**RUNNING $\alpha_s$ IN QCD AT STRONG B**

- In deep UV region $\alpha_s$ is not affected by B-field

\[ M_g^2 \approx \frac{\alpha_s}{\pi} \sum |e_f B| \]

\[ m_{dyn}^2 \ll |k_\parallel|^2 \ll |eB| \]

\[ \frac{1}{\alpha_s} \approx b \ln \frac{|eB|}{\Lambda_{QCD}^2} \]

\[ \lambda_{QCD} \ll \Lambda_{QCD} \]

\[ \lambda_{QCD} \ll m_{dyn} \ll \sqrt{|eB|} \]

\[ \lambda_{QCD} \ll \Lambda_{QCD} \]

\[ \lambda_{QCD} \ll m_{dyn} \ll \sqrt{|eB|} \]

\[ \lambda_{QCD} \ll \Lambda_{QCD} \]

\[ \lambda_{QCD} \ll m_{dyn} \ll \sqrt{|eB|} \]
The general form of the equation is similar to that in QED

\[ G^{-1}(x, y) = G_0^{-1}(x, y) + 4\pi\alpha_s \gamma^\mu T^A G(x, y) \gamma^\nu T^B D^{AB}_{\mu\nu} (y - x) \]

Note that the inverse propagator \( G^{-1}(x, y) \) has the same (!) Schwinger phase as \( G(x, y) \)

- Non-Abelian structure of the theory (\( T^A T^A = C_2 \)): \( \alpha \to \frac{N_c^2 - 1}{2N_c} \alpha_s \)

- Screening effects are included via the polarization function

\[ P^{AB}_{\mu\nu} (x - y) = 4\pi\alpha_s \text{tr}[\gamma_\mu T^A \tilde{G}(x - y) \gamma_\nu \lambda T^A \tilde{G}(y - x)] \]

- Similar to QED, in the strong field limit (\( \sqrt{|eB|} \gg \Lambda_{QCD} \))

\[ P^{AB,\mu\nu}_{\mu\nu} \sim \frac{\alpha_s}{6\pi} \delta^{AB} \left(k^\mu_k^\nu - k^2 g^{\mu\nu} \right) \sum_{q=1}^{N_f} \frac{|e_q B|}{m_q^2}, \quad \text{for } |k^2| \ll m_q^2 \]

\[ P^{AB,\mu\nu}_{\mu\nu} \sim -\frac{\alpha_s}{\pi} \delta^{AB} \left(k^\mu_k^\nu - k^2 g^{\mu\nu} \right) \sum_{q=1}^{N_f} \frac{|e_q B|}{k^2}, \quad \text{for } m_q^2 \ll |k^2| \ll |eB| \]
**SCREENING MASSES: LATTICE**

- Electric and magnetic screening masses on the lattice are fitted well by [Bonati et al., Phys. Rev. D 95, 074515 (2017)]

\[
\frac{m_E^d}{T} = a_E^d \left[ 1 + c_{1;E}^d \frac{|e|B}{T^2} \arctan\left( \frac{c_{2;E}^d}{c_{1;E}^d} \frac{|e|B}{T^2} \right) \right]
\]

(and similar for the magnetic one)
**Expression for Dynamical Mass**

- In the region $m_{dyn}^2 \ll |k|^2 \ll |eB|$, which is most relevant for the fermion-pairing dynamics, the gluon has a “mass”

$$M_g^2 \simeq \frac{\alpha_s}{\pi} \sum_f |e_f B| = \frac{\alpha_s}{3\pi} (2N_u + N_d) |eB|$$

- As in QED, in order to tame singular infrared corrections in higher-order diagrams, a special non-local gauge is assumed for the gluon propagator

- Up to replacements $\alpha \rightarrow \frac{N_c^2 - 1}{2N_c} \alpha_s$ and $M_\gamma^2 \rightarrow M_g^2$, the gap equation looks as in QED. Thus,

$$m_q^2 \simeq 2C_1 |e_q B| \left(c_q \alpha_s\right)^{2/3} \exp \left[-\frac{4N_c\pi}{\alpha_s(N_c^2 - 1) \ln(C_2/c_q \alpha_s)}\right]$$

where $C_1 \simeq C_2 \simeq 1$ and $c_q \simeq (2N_u + N_d)|e|/(6\pi|e_q|)$
Quantitatively, dynamical masses are \((\sqrt{|eB|} \gg \Lambda_{\text{QCD}})\).
CHIRAL CONDENSATE IN LATTICE QCD

\[ \Delta(\Sigma_u + \Sigma_u) / 2 \]

\[ \Sigma_f = \bar{\psi}_f \psi_f \propto m_{dyn,f} \]

\[ T=0 \]

Lecture #4
Magnetic Catalysis in QCD

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Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports 576 (2015)
**Nambu-Goldstone Bosons (Pions)**

- Original global chiral symmetry
  \[ SU_L(N_u) \times SU_R(N_u) \times SU_L(N_d) \times SU_R(N_d) \times U_A^{-1} \]
  breaks down to
  \[ SU_V(N_u) \times SU_V(N_d) \]
- A total number of broken-symmetry generators: \( N_u^2 + N_d^2 - 1 \)
- Thus, there should be \( (N_u^2 + N_d^2 - 1) \) massless NG bosons
- The unitary pion fields can be written in terms of the coset space generators
  \[ \Sigma_u \equiv \exp \left( i \sum_{A=1}^{N_u^2-1} \lambda^A \pi^A_u / f_u \right), \quad \Sigma_d \equiv \exp \left( i \sum_{A=1}^{N_d^2-1} \lambda^A \pi^A_d / f_d \right) \]
  and \[ \tilde{\Sigma} \equiv \exp \left( i \sqrt{2} \tilde{\pi} / \tilde{f} \right) \]
- In a very strong magnetic field another light pseudo-NG boson, associated with anomalous \( U_A(1) \), may appear
NAMBU-GOLDSTONE BOSONS (PIONS)

- The low-energy effective action should have the form

\[ \mathcal{L}_{NG} \simeq \frac{f_u^2}{4} \text{tr} \left( g_\|^{\mu\nu} \partial_\mu \Sigma_u \partial_\nu \Sigma_u^\dagger + v_u^2 g_\perp^{\mu\nu} \partial_\mu \Sigma_u \partial_\nu \Sigma_u^\dagger \right) + \ldots \]

- The pion decay constants are defined by

\[
\begin{align*}
\langle 0 \left| \bar{\psi} \gamma^\mu \gamma^5 \frac{\lambda^A}{2} \psi \right| \pi^B (P) \rangle &= P^\mu f_\pi \delta^{AB} = -i \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left( \gamma^\mu \gamma^5 \frac{\lambda^A}{2} \chi^B_q (k, P) \right)
\end{align*}
\]

where \( P^\mu = (P^0, v_\perp \vec{P}_\perp, P^3) \)

- The Bethe-Salpeter wave function looks like in QED in LLL approximation. So, we find that \( v_\perp \approx 0 \), and

\[
f_q^2 = 4N_c \int \frac{d^2 k_\perp d^2 k_\|}{(2\pi)^4} \exp \left( -\frac{k_\perp^2}{|e_q B|} \right) \frac{m_q^2}{(k_\|^2 + m_q^2)^2}
\]

which can be easily calculated, giving

\[
f_u^2 = \frac{N_c |eB|}{6\pi^2} \quad \text{and} \quad f_d^2 = \frac{N_c |eB|}{12\pi^2}
\]
**LOW-ENERGY REGION,** $|k^2_\parallel| \approx m^2_{\text{dyn}}$

- Massive quarks decouple from the low-energy dynamics

- Gluons are the only “light” degrees of freedom

- Assuming that $\Lambda^2_{QCD} \ll m^2_{\text{dyn}}$, the gluodynamics has a semi-perturbative region, $|k^2_\parallel| \approx m^2_{\text{dyn}}$, where

  $$\frac{1}{\tilde{\alpha}_s(\mu)} - \frac{1}{\alpha_s} \approx b_0 \ln \frac{\mu^2}{m^2_{\text{dyn}}}$$

  here $b_0 = \frac{11 N_c}{12\pi}$ and $\frac{1}{\alpha_s} \approx b \ln \frac{|eB|}{\Lambda^2_{QCD}}$ (Recall: $b = \frac{11 N_c - 2N_f}{12\pi}$)

- Then, we find that the new confinement scale where $\tilde{\alpha}_s = \infty$:

  $$-b \ln \frac{|eB|}{\Lambda^2_{QCD}} \approx b_0 \ln \frac{\lambda^2_{QCD}}{m^2_{\text{dyn}}} \Rightarrow \lambda_{QCD} = m_{\text{dyn}} \left( \frac{\Lambda_{QCD}}{\sqrt{|eB|}} \right)^{b/b_0}$$
**Low-energy gluodynamics**

- Quadratic part of low-energy effective action for gluons

\[
\mathcal{L}_{\text{glue, eff}}^{(2)} = -\frac{1}{2} \sum_{A=1}^{N_c^2-1} A^A_\mu (-k) \left[ g^{\mu\nu} k^2 - k^\mu k^\nu + \kappa \left( g^{\parallel\parallel} k^2 - k^\parallel k^\parallel \right) \right] A^A_\nu (k)
\]

where the susceptibility \( \kappa \) is extracted from the polarization tensor \( \mathcal{P}^{AB}_{\mu\nu} \) in the region \( |k^\parallel| \ll m_{\text{dyn}}^2 \), i.e.,

\[
\kappa = \frac{\alpha_s}{6\pi} \sum_{q=1}^{N_f} \frac{|e_q B|}{m_q^2} = \frac{1}{12C_1\pi} \sum_{q=1}^{N_f} \left( \frac{\alpha_s}{c_q^2} \right)^{1/3} \exp \left( \frac{4N_c\pi}{\alpha_s(N_c^2 - 1) \ln(C_2/c_q\alpha_s)} \right) \gg 1
\]

- The requirement of gauge invariance allows to write down the complete expression for the gluon action

\[
\mathcal{L}_{\text{glue, eff}} \simeq \frac{1}{2} \sum_{A=1}^{N_c^2-1} \left( \mathbf{E}^A_\perp \cdot \mathbf{E}^A_\perp + \epsilon E^A_3 E^A_3 - \mathbf{B}^A_\perp \cdot \mathbf{B}^A_\perp - B^A_3 B^A_3 \right)
\]

where \( \epsilon = 1 + \kappa \) is a chromo-dielectric constant (note \( \epsilon \gg 1 \)),

\( E^A_i = F^A_{0i} \) and \( B^A_i = \frac{1}{2} \varepsilon_{ijk} F^A_{jk} \) are chromo-fields.
By using the guidance from an analogous anisotropic QED, the static potential between a pair of quarks should be given by

\[ V(x, y, z) \approx \frac{g_s^2}{4\pi \sqrt{z^2 + \epsilon (x^2 + y^2)}} \]

which is valid for a range of distance scales \(m_{dyn}^{-1} \approx r \approx \lambda_{QCD}^{-1}\).

Note that the effective coupling constants

\[ \alpha_s^{\parallel} = \frac{g_s^2}{4\pi v_g^{\parallel}} \approx \frac{g_s^2}{4\pi}, \quad \text{where} \quad v_g^{\parallel} \approx 1 \]

\[ \alpha_s^{\perp} = \frac{g_s^2}{4\pi \sqrt{\epsilon} v_g^{\perp}} \approx \frac{g_s^2}{4\pi}, \quad \text{where} \quad v_g^{\perp} \approx 1/\sqrt{\epsilon} \]

are approximately the same in all directions.

A posteriori, this naïve “isotropy” may justifies the use of running behavior as in isotropic gluodynamics (not rigorous).
**POTENTIAL ON LATTICE**

- Quark-antiquark potential was fitted by Cornell potential,
  \[ V(r) = -\frac{\alpha}{r} + \sigma r + V_0 \]
- where \( \sigma \) is the string tension and \( \alpha \) is the Coulombic coefficient

**ANISOTROPY IN DETAIL**

- The dependence of the potential as a function of angle $\theta$ between $\vec{B}$ and $q\bar{q}$ orientation [Bonati et al., Phys. Rev. D 94, 094007 (2016)]

$$V(r, \theta; B) = -\frac{\alpha(\theta; B)}{r} + \sigma(\theta; B)r + V_0(\theta; B)$$

- With increasing angle $\theta$, the string tension increases.
What to expect at nonzero temperature (in strong B limit)?

- Very low temperatures, $T \ll \lambda_{QCD}$
  - Ground state in not affected much
  - Color is confined, lowest energy states are glueballs
  - Chiral symmetry is broken ($T \ll \lambda_{QCD} \ll m_{dyn}$)

- Intermediate temperatures, $\lambda_{QCD} \ll T \ll m_{dyn}$
  - Color is deconfined; gluons are thermally populated
  - Chiral symmetry is still broken ($\lambda_{QCD} \ll T \ll m_{dyn}$)

- Moderately high temperatures, $m_{dyn} \ll T \ll \sqrt{|eB|}$
  - Chiral symmetry is restored ($m_{dyn} \ll T$)
**Predicted Phase Diagram**

![Graph showing the predicted phase diagram for QCD with critical temperature $T_c$ and magnetic field $eB$ parameters.](image)

Inverse Catalysis at $T \neq 0$

$\Delta(\Sigma_u + \Sigma_d) / 2$

$T = 0$
$T = 130 \text{ MeV}$
$T = 148 \text{ MeV}$
$T = 153 \text{ MeV}$
$T = 163 \text{ MeV}$
$T = 176 \text{ MeV}$

$eB (\text{GeV}^2)$

• The temperature dependence at several fixed values of $B$

\[
\Sigma_{\text{fit}} = \frac{\Sigma_0}{\exp \left( \frac{T-T_c}{\Delta T} \right) + 1}
\]

• Confinement strongly affects the low-temperature region
DEPENDENCE OF $T_C$ VS. $B$

• Gluon screening (?)
• Polyakov loops (?)

[Bruckmann, G. Endrodi, T. G. Kovacs, JHEP 04 (2013) 112]
SUPER-STRONG $B$: PREDICTION

![Graph showing deconfinement transition line, prediction, crossover, critical endpoint, and first order transition.]

PREDICTED PHASE DIAGRAM

\[ T_c \sim m_{\text{dyn}} \text{ (chiral symmetry restoration)} \]

\[ T_c^* \sim \lambda_{\text{QCD}} \text{ (deconfinement)} \]