HYDRO22 Colloquium:

Relativistic-like electron hydrodynamics in Dirac semimetals

Igor Shovkovy

Emergent Hydrodynamics in Condensed Matter and High-Energy Physics
Chiral plasma

- **Heavy-ion collisions** *(high temperature)*

- **Super-dense matter in compact stars** *(high density)*
  [Effects seem negligible due slow chirality production and a large chirality flip rate]

- **Early Universe** *(high temperature)*
  [Joyce & Shaposhnikov, PRL 79, 1193 (1997)]

- **Magnetospheres of magnetars**
  (electron-positron plasma at moderately high temperature)

- **Electron plasma in Dirac/Weyl (semi-)metals**
  [Gorbar, Miransky, Shovkovy, Sukhachov, Electronic Properties of Dirac and Weyl Semimetals (World Scientific, Singapore, 2021)]

- **Other:** cold atoms, superfluid $^3$He-A, etc.
Chirality

- Only *massless* Dirac/Weyl fermions have a well-defined chirality \((\gamma^5 \psi = \pm \psi)\)*:
  - **Right-handed** (spin parallel to momentum)
  - **Left-handed** (spin opposite to momentum)

- The chirality of *massive* Dirac fermions is *almost* well-defined in the *ultra-relativistic* regime*
  - High temperature: \(T \gg m\)
  - High density: \(\mu \gg m\)

*Note: like the particle spin, chirality is a quantum property*
Anomalous chiral matter

• Relativistic matter made of chiral fermions may allow $n_L \neq n_R$ existing on *macroscopic* time/distance scales

• The spacetime dynamics of $n = n_R + n_L$ and $n_5 = n_R - n_L$ is governed by continuity equations

$$\frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\frac{\partial n_5}{\partial t} + \nabla \cdot \vec{j}_5 = \frac{e^2 \vec{E} \cdot \vec{B}}{2\pi^2} - \Gamma_m n_5$$

where a nonzero chirality flip rate is accounted by $\Gamma_m$

• Chiral anomaly can produce *macroscopic* effects in plasmas
Anomalous effects

- **Theory**: Many *macroscopic* chiral anomalous effects were proposed (chiral magnetic effect, chiral separation effect, etc.)

- **High-energy physics**
  - Charged particle correlations
  - Eccentricities of particle flows
  - New types of collective modes
  - Inverse magnetic cascade
  - ...

- **Condensed matter physics**
  - Negative longitudinal magnetoresistance
  - Nonlocal anomalous transport
  - Chiral charge pumping
  - ...

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DIRAC & WEYL SEMIMETALS

Real band structures

- Electron quasiparticles with a wide range of properties are possible
- They may even have an emergent spinor structure of Dirac/Weyl fermions, e.g.,

\[ H_W \approx v_F (\vec{k} \cdot \vec{\sigma}) \]

where \( \vec{k} \) is the momentum measured from the Weyl node and \( v_F \) is the Fermi velocity
- How common is this?

Relativistic-like band crossing

Do energy levels cross? Or do they repel?

A generic 2-band Hamiltonian reads

\[ H_\mathbf{k} = a_\mathbf{k} + \mathbf{b}_\mathbf{k} \cdot \mathbf{\sigma} \quad \Rightarrow \quad E_\mathbf{k} = a_\mathbf{k} \pm \sqrt{(\mathbf{b}_\mathbf{k})^2} \]

The bands cross when \[ \mathbf{b}_\mathbf{k} = 0 \]

These 3 equations can be solved by adjusting \( \mathbf{k} \) in 3D

[\text{Witten, Riv. Nuovo Cimento 39, 313 (2016)}]
Emergent chirality in solids

Near a band crossing (e.g., $\vec{k} \approx \vec{k}_+$)

$$H_k = a_{k+} + \left( \nabla_k a_\vec{k} \cdot \delta \vec{k} \right) + \sum_{i,j} \sigma_i b_{ij} \delta k_j$$

cone tilting

After an orthogonal transformation

$$b_{ij} \equiv \partial b_i / \partial k_j \rightarrow \hbar \nu_i \delta_{ij}$$

Assuming isotropy & a suitable reference point,

$$H_k = \pm \nu_F (\vec{\sigma} \cdot \vec{k})$$

which is the Weyl Hamiltonian

The chirality is defined by

$$\lambda = \text{sign}[\det(b_{ij})]$$
Weyl quasiparticles

- The quasiparticle eigenstates for Weyl Hamiltonian $H_\lambda = \lambda \nu_F (\vec{k} \cdot \vec{\sigma})$ are

$$\psi^\lambda_k = \frac{1}{\sqrt{2} \sqrt{\epsilon_k^2 + \lambda \nu_F \epsilon_k k_z}} \begin{pmatrix} \nu_F k_z + \lambda \epsilon_k \\ \nu_F k_x + i \nu_F k_y \end{pmatrix}$$

- The quasiparticle energy $\epsilon_k = \nu_F \sqrt{k_x^2 + k_y^2 + k_z^2}$ is relativistic-like

- Mapping $k \to \psi^\lambda_k$ has a nontrivial topology

- Consider adiabatic evolution of the wave function from $\psi_k$ to $\psi_{k+\delta k}$:

$$\langle \psi_k | \psi_{k+\delta k} \rangle \approx 1 + \delta k \cdot \langle \psi_k | \nabla_k | \psi_k \rangle \approx e^{i \lambda_k \cdot \delta k}$$

where $\lambda_k = -i \langle \psi_k | \nabla_k | \psi_k \rangle$ is the Berry connection
For Weyl eigenstates, the Berry curvature is

$$\Omega_k \equiv \nabla_k \times a_k = \lambda \frac{\vec{k}}{2k^3}$$

The Chern number (topological charge)

$$C = \frac{1}{2\pi} \int \Omega_k \cdot dS_k = \frac{\lambda}{2\pi} \int \frac{\vec{k}}{2k^3} \cdot \frac{\vec{k}}{k} k^2 \sin \theta \, d\theta d\phi = \lambda$$

In a solid state material, the Brillouin zone is compact.

A closed surface around a node at $\vec{k}_0$ is also a closed surface around the rest of the Brillouin zone.

Thus, Weyl fermions come in pairs of opposite chirality.


[Morimoto & Nagaosa, Scientific Reports 6, 19853 (2016)]
Idealized Dirac and Weyl model

- Low-energy Hamiltonians of a Dirac and Weyl materials

\[
H = \int d^3 r \, \bar{\psi} \left[ -i v_F (\vec{\gamma} \cdot \vec{p}) - (\vec{b} \cdot \vec{\gamma}) \gamma^5 + b_0 \gamma^0 \gamma^5 \right] \psi
\]

Dirac (e.g., Na\textsubscript{3}Bi, Cd\textsubscript{3}As\textsubscript{2}, ZrTe\textsubscript{5})

Weyl (e.g., TaAs, NbAs, TaP, NbP, WTe\textsubscript{2})
ELECTRON HYDRODYNAMICS
Electron hydrodynamics

- First ideas date back to the 1960s [Radii Gurzhi, JETP 17, 521 (1963)]

Ballistic:
\[ l_p \gg l_{ee} \gg L \quad R \sim L^{-1} \]

Hydro:
\[ l_{ee} \ll L \ll l_p \quad R \sim \frac{l_{ee}}{L^2} \sim T^{-5} L^{-2} \]

Hydro + impurities:
\[ l_p \ll L^2/l_{ee} \quad R \sim l_p \]

Ohmic \((l_{ph} \ll l_p)\): \[ R \sim T^5 \]
Higher than ballistic transport

[Guo et al., PNAS USA 114, 3068 (2017)]

[Guo et al., PNAS USA 114, 3068 (2017)]

[Kumar et al., Nat. Phys. 13, 1182 (2017)]

Other Signatures:

• Negative nonlocal resistance
  [Bandurin et al., Science 351, 1055 (2016)]
  [Levitov & Falkovich, Nat. Phys. 12, 672 (2016)]

• Visualization of the Poiseuille flow
  [Sulpizio et al., Nature 576, 75 (2019)]
Hydrodynamics in Weyl semimetals


$$\rho = \rho_0 + \rho_1 w^\beta$$

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RELATIVISTIC-LIKE ELECTRON HYDRODYNAMICS

[Gorbar, Miransky, Shovkovy, Sukhachov, Phys. Rev. B 97, 121105(R) (2018)]
Chiral Hydrodynamics (plasma)

- Evolution of conserved quantities:

\[
\frac{\partial T^0}{\partial t} + \frac{\partial T^j}{\partial x^i} = -enE^j - e\left(\vec{j} \times \vec{B}\right)^j + F_{\text{other}}^j
\]

\[
\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = -e\left(\vec{E} \cdot \vec{j}\right) + W_{\text{other}}
\]

\[
\frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0
\]

\[
\frac{\partial n_5}{\partial t} + \nabla \cdot \vec{j}_5 = -\frac{e^2 (\vec{B} \cdot \vec{E})}{2\pi^2} - \Gamma_5 n_5
\]

⊠ Maxwell equations

Note:
\[
T^{00} = \varepsilon + \cdots
\]
\[
T^{0i} = w\nu^i + \cdots
\]
\[
T^{ij} = w\nu^i\nu^j - P\delta^{ij} + \cdots
\]
\[
w = \varepsilon + P
\]
Constitutive relations

- Expressions for currents and $T^{\mu\nu}$

\[
\vec{j} = n\vec{v} + \vec{j}_a + \vec{j}_{\text{dis}}
\]

\[
\vec{j}_5 = n_5\vec{v} + \vec{j}_{5,a} + \vec{j}_{5,\text{dis}}
\]

\[
T^{\mu\nu} = \varepsilon v^\mu v^\nu - \Delta^{\mu\nu}P + h^\mu v^\nu + v^\mu h^\nu + \tau^{\mu\nu}_{\text{dis}}
\]

- Anomalous terms:

\[
\vec{j}_a = \vec{j}_{\text{CS}} + \sigma_\omega \vec{\omega} + \sigma_B \vec{B} \quad \& \quad \vec{j}_{5,a} = \vec{j}_{5,\text{CS}} + \sigma_\omega^5 \vec{\omega} + \sigma_B^5 \vec{B}
\]

where

\[
\sigma_\omega = \frac{\mu \mu_5}{\pi^2 \hbar^2}, \quad \sigma_B = \frac{e \mu_5}{2\pi^2 \hbar^2}
\]

\[
\sigma_\omega^5 = \frac{3(\mu^2 + \mu_5^2) + \pi^2 T^2}{6\pi^2 \hbar^2}, \quad \sigma_B^5 = \frac{e \mu}{2\pi^2 \hbar^2}
\]
Hydrodynamics in Weyl metals

• The Euler equation from CKT:

\[
\frac{1}{v_F} \partial_t \left( \frac{\epsilon + P}{v_F} u + \sigma^{(\epsilon, B)} B \right) = -en \left( E + \frac{1}{c} [u \times B] \right) + \frac{\sigma^{(B)}(E \cdot B)}{3v_F^2} u - \frac{\epsilon + P}{\tau v_F^2} u + O(\nabla r)
\]

• The energy conservation from CKT

\[
\partial_t \epsilon = -E \cdot (enu - \sigma^{(B)} B) + O(\nabla r)
\]

⊕ Maxwell equations

• One must include topological Chern-Simons currents and densities,

\[
\rho_{CS} = -\frac{e^3 (b \cdot B)}{2\pi^2 \hbar^2 c^2}
\]

\[
\mathbf{J}_{CS} = -\frac{e^3 b_0 B}{2\pi^2 \hbar^2 c} + \frac{e^3 [b \times \mathbf{E}]}{2\pi^2 \hbar^2 c}
\]

ANOMALOUS HYDRO MODES

Collective modes

- Local values of $\delta \mu$, $\delta \mu_5$, $\delta T$, $\delta \tilde{u}$, etc. oscillate.
- Seek solutions of linearized equations as plane waves,
  \[ \delta \mu = \delta \mu_0 \exp(-i\omega t + ik \cdot \vec{r}), \]
  etc.
- Account for all constitutive relations, e.g.,
  \[
  \begin{align*}
  \delta \rho &= -e \delta n + \text{ anomalous terms} \\
  \delta \rho_5 &= -e \delta n_5 + \text{ anomalous terms} \\
  \delta J &= -en_0 \delta u + \text{ anomalous terms} \\
  \delta J_5 &= -en_{5,0} \delta u + \text{ anomalous terms}
  \end{align*}
  \]
Rich spectrum of hydro modes

- One example: longitudinal anomalous Hall wave (with $\mathbf{k} \parallel \mathbf{B}_0$ and $\mathbf{b} \perp \mathbf{B}_0$):

$$
\omega_{\text{LAHW}, \pm} \approx \pm \frac{\hbar B_0 k_\parallel \sqrt{3 v_F^3 \left( \pi^3 c^4 \hbar T_0^2 + 3 e^4 \mu_m v_F^3 b_\perp^2 \right)}}{c T_0 \sqrt{\pi^3 \mu_m \left( 3 \varepsilon_e v_F^3 \hbar^3 B_0^2 + 4 \pi e^2 T_0^2 b_\perp^2 \right)}} + O(k_\parallel^3)
$$

\begin{align*}
\text{n continuity equation} & \quad \frac{T^2 \omega}{3 v_F^3 \hbar} \delta \mu + \frac{eB_0 k_\parallel}{2 \pi^2 c} \delta \mu_5 = 0 \\
\text{n}_5 \text{ continuity equation} & \quad \frac{eB_0 k_\parallel}{2 \pi^2 c} \delta \mu + \frac{T^2 \omega}{3 v_F^3 \hbar} \delta \mu_5 - i \frac{e^2 B_0}{2 \pi^2 c} \delta E_\parallel = 0 \\
\text{Maxwell's equations} & \quad \left( \omega^2 - \frac{c^2 k_\parallel^2}{\varepsilon_e \mu_m} \right) \delta \tilde{E}_\perp - i \frac{2 e^3 \omega b_\perp}{\pi c \varepsilon_e \hbar^2} \delta E_\parallel = 0
\end{align*}

ENTROPY WAVE INSTABILITY

[Sukhachov, Gorbar, Shovkovy, Phys. Rev. Lett. 127, 176602 (2021)]
Dyakonov-Shur instability

[Dyakonov, Shur, Phys. Rev. Lett. 71, 2465 (1993)]

\[ J_0 = -enu_0 \]
\[ \partial_t u + u\partial_x u = -\frac{e}{m}\partial_x \varphi \]
\[ \partial_t \varphi + \partial_x (\varphi u) = 0 \]

Gradual channel approximation:

\[ n \propto \varphi \]
\[ \delta n \sim e^{-i\omega t} \]

\[ \text{Re} \left[ \omega \right] = \frac{|v_p^2 - u_0^2|}{v_p} \frac{\pi l}{2L} \]
\[ \text{Im} \left[ \omega \right] = \frac{|v_p^2 - u_0^2|}{2Lv_p} \ln \left| \frac{v_p + u_0}{v_p - u_0} \right| \]

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Observation & application

[Emitter] Emitter

[Detector] Detector

\[ \omega \sim \text{THz} \]

Experimental observation:

- \( \times \) Emission
- \( \checkmark \) Detection

[Vitiello, et al., Nano Lett. 12, 96 (2012)]
[Vicarelli, et al., Nat. Mater. 11, 865 (2012)]

[Vitiello, et al., Nano Lett. 12, 96 (2012)]
Relativistic-like case

• System of equations

\[
\frac{1}{v_F^2} \left[ \partial_t + (u \cdot \nabla) \right] (uw) + \frac{1}{v_F^2} wu(\nabla \cdot u) = -\nabla P + en \nabla \varphi + \eta \Delta u + \frac{\eta}{d} \nabla (\nabla \cdot u) - \frac{wu}{v_F^2},
\]

\[
-e \partial_t n + (\nabla \cdot J) = 0,
\]

\[
\partial_t \epsilon + (\nabla \cdot J^\epsilon) = (E \cdot J),
\]

\[
\Delta \varphi = 4\pi e (n - n_0).
\]

• Collective modes in an infinite system:

3D: \( \omega_\pm \approx \pm \sqrt{\omega_p^2 + v_s^2 k_x^2} + \frac{2}{3} u_0 k_x \),

2D: \( \omega_\pm \approx \pm v_p k_x + \frac{1}{2} u_0 k_x \),

2D and 3D: \( \omega_e \approx u_0 k_x \).

where \( v_s = v_F / \sqrt{d} \) and \( \omega_p^2 = 4\pi e^2 n_0^2 v_F^2 / w_0 \).
Instability in 3D: analytical results

- Boundary conditions (\(\sim\) Dyakonov-Shur BC):
  
  \[
  \begin{align*}
    n_1(x = 0) &= 0, \\
    J_x(x = L) &\equiv n_0 u_1(x = L) + u_0 n_1(x = L) = 0, \\
    T_1(x = 0) &= 0.
  \end{align*}
  \]

- Frequencies of the collective modes:

  \[
  \begin{align*}
    \omega^{3D}_{\pm} &\approx \pm \sqrt{\omega_p^2 + \left( \frac{v_s \pi}{L} \left( l + \frac{1}{2} \right) \right)^2 + \frac{2u_0}{3L} (3 - 2\Lambda_p^2)} \\
    \omega^{3D}_e &\approx \frac{2\pi l}{L} u_0 - i \frac{u_0 \omega_p}{v_s} - i \frac{u_0}{L} \ln \left[ \frac{3}{8} \frac{v_s^2}{u_0^2 (1 - \Lambda_p^2)} \right], \quad l = 0, \pm 1, \pm 2, \ldots \\
    \Lambda_p &= \omega_p / (v_s q_{TF}) < 1, \quad \lim_{T \to 0} \Lambda_p = 1.
  \end{align*}
  \]

Plasmon instability

Entropy wave instability
Instability in 3D (numerical)

- Plasmon modes: $\text{Re}(\omega_{DS}) \approx \omega_P$
- Entropy waves: $\text{Re}(\omega_{EW}) \propto u_0$
- $\text{Im}(\omega_{EW}) \gg \text{Im}(\omega_{DS})$
- DSI and EWI occur for opposite sign($u_0$)
Instability in 2D (numerical)

\[ \omega_{\pm}^{2D} \approx \pm v_p \frac{\pi}{L} \left( l + \frac{1}{2} \right) + i \frac{u_0}{2L} (4 - 3 \Lambda_p^2) \]

\[ \omega_e^{2D} \approx \frac{2\pi l}{L} u_0 - i \frac{u_0}{L} \ln \left[ \frac{2}{3} \frac{v_p^2}{u_0^2 (1 - \Lambda_p^2)} \right] \]

Summary

• Electron hydrodynamics in Dirac/Weyl semimetals is chiral (if realized)

• Chern-Simon currents/densities appear and play role
  [Gorbar, Miransky, Shovkovy, Sukhachov, Phys. Rev. B 97, 121105(R) (2018)]

• New anomalous hydrodynamic modes are expected

• Convection is impossible due to strong Coulomb effects (3D) and impurities (2D)
  [Sukhachov, Gorbar, Shovkovy, Phys. Rev. B 104, 121113 (2021)]

• Entropy wave instability can develop (signature of relativistic-like nature)
  [Sukhachov, Gorbar, Shovkovy, Phys. Rev. Lett. 127, 176602 (2021)]