DIMENSIONAL REDUCTION & CATALYSIS OF DYNAMICAL SYMMETRY BREAKING BY A MAGNETIC FIELD

Igor Shovkovy
Arizona State University

Physics Opportunities at a Lepton Collider in the Fully Nonperturbative QED Regime
QCD in Magnetic Fields

• Relativistic collisions of *heavy ions* produce quark-gluon plasma & strong magnetic fields

\[ 10^{18} - 10^{19} \text{ Gauss} \ (\sqrt{|eB|} \sim 100 \text{ MeV}) \]

• Quark matter may form inside *magnetars*

\[ 10^{14} - 10^{16} \text{ Gauss} \ (\sqrt{|eB|} \sim 1 \text{ MeV to } 10 \text{ MeV}) \]

• Strong magnetic field is a *theoretical tool* to probe the confinement dynamics in QCD at short distance scales, \( \ell \sim 1/\sqrt{|eB|} \)

\[ \geq 10^{19} \text{ Gauss} \ (\sqrt{|eB|} \geq 240 \text{ MeV}) \]
SET THE STAGE

• Lagrangian density of QCD in an external magnetic field

\[ \mathcal{L} = -\frac{1}{2} F_{A}^{\mu \nu} F_{\mu \nu}^{A} + \bar{\psi}_f (i \gamma^\mu D_\mu) \psi_f \]

where \( D_\mu = \partial_\mu + ig A_\mu^A \lambda^A / 2 + ie f A_\mu^\text{ext} \)

\[ F_{\mu \nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f^{ABC} A_\mu^B A_\nu^C \]

• The global chiral symmetry of the model

\[ SU_L(N_u) \times SU_R(N_u) \times SU_L(N_d) \times SU_R(N_d) \times U_A^{-}(1) \]

chiral symmetry of up-flavors
chiral symmetry of down-flavors
anomaly-free combination of \( U_A^{(u)}(1) \) and \( U_A^{(d)}(1) \)

• Quark masses \( m_u \neq m_d \neq 0 \) break the symmetry down to

\[ SU_V(N_u) \times SU_V(N_d) \]
RUNNING COUPLING & CONFINEMENT

- Coupling constant in QCD runs with the energy scale,

\[
\frac{1}{\alpha_s(\mu)} \simeq b \ln \frac{\mu^2}{\Lambda_{QCD}^2}, \quad \text{where} \quad b = \frac{11 N_c - 2 N_f}{12 \pi}
\]

- The question is: What happens in a strong magnetic field?
**Free Dirac Fermions, B≠0**

- Dirac equation:
  \[(i\gamma^\mu D_\mu - m)\psi = 0\]
  where \(A_\mu = (A_0, -\vec{A})\) and the Landau gauge \(\vec{A} = (-By, 0,0)\) is used.

- Solutions take the form \(\psi = (i\gamma^\mu D_\mu + m)\phi\), where
  \[\phi_{k,\pm} \propto \frac{1 \pm iy^1y^2}{2} \varphi_k(y)e^{-i\omega t + ip_x x + ip_z z}\]

- Here \(\varphi_k\) are harmonic oscillator wave functions,
  \[\varphi_k \propto H_k(\xi)e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l \quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}\]

- The Landau level energies are
  \[\omega = E_n^\pm = \pm \sqrt{2n|eB| + p_z^2 + m^2}\]
  where \(n = k + \frac{1}{2} + s_z\) and \(s_z = \pm \frac{1}{2}\) is an eigenvalue of \(\frac{i}{2} \gamma^1 \gamma^2\).
Landau energy levels at $m = 0$

\[ E_n^\pm = \pm \sqrt{2n|eB|} + p_z^2 \]

where $n = k + \frac{1}{2} + s_z$

- Lowest Landau level is spin polarized

\[ E_0^\pm = \pm p_z \quad (k = 0, \ s_z = -\frac{1}{2}) \]

- Higher Landau levels ($n \geq 1$) are twice as degenerate:

  (i) $k = n$ \& $s = -\frac{1}{2}$

  (ii) $k = n - 1$ \& $s = +\frac{1}{2}$
The Landau level energies are independent of $p_x$

$$E_n^\pm = \pm \sqrt{2n|eB| + p_z^2 + m^2}$$

Consider a finite box with periodic boundary conditions

The wave function $\psi(x) \propto e^{ip_xx}$ satisfies $\psi(0) = \psi(L_x)$, i.e.,

$$e^{ip_xL_x} = 1 \implies p_x = \frac{2\pi n}{L_x}, \quad n = 1, 2, ..., N_{\text{max}}$$

Since $p_x$ defines the center of Landau orbits in $y$-direction:

$$y_{c,\text{max}} \approx -p_{x,\text{max}}l^2 \lesssim L_y \implies \frac{2\pi N_{\text{max}}}{L_x} \frac{1}{|eB|} \approx L_y$$

Thus, the degeneracy is

$$N_{\text{max}} \approx \frac{|eB|}{2\pi} L_x L_y$$
**DIRAC PROPAGATOR AT $B\neq 0$**

- By definition,

$$G(r, r') = i \left< r \left| (i \gamma^\mu D_\mu - m) \right|^{-1} r' \right>$$

$$= i (i \gamma^\mu D_\mu + m) \left< r \left| \left[ -D^\mu D_\mu + i \gamma^1 \gamma^2 eB - m^2 \right] \right|^{-1} r' \right>$$

$$= i (i \gamma^\mu D_\mu + m) \sum \left< r \left| k, p_z, s_z \right> (\omega^2 - E_n^2)^{-1} \left< k, p_z, s_z \left| r' \right> \right>$$

- Note that the explicit form of the wave functions is the same as before

$$\psi_{k,p_z,s_z}(r) = \left< r \left| k, p_z, s_z \right> \propto H_k(\xi) e^{-\frac{\xi^2}{2}} e^{-i\omega t + ip_z z} U_{s_z}, \text{ where } \xi = \frac{\gamma}{l} + p_x l$$

- Then, the propagator has the form

$$G(\omega, p_z; \vec{r}_\perp, \vec{r}'_\perp) = e^{i\Phi(\vec{r}_\perp, \vec{r}'_\perp)} \tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}'_\perp)$$

where $\Phi(\vec{r}_\perp, \vec{r}'_\perp) = -e \int_{\vec{r}'_\perp}^{\vec{r}_\perp} A_\nu dr^\nu$ is the Schwinger phase, and

$$\tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}'_\perp) = \int \frac{d^2 p_\perp}{(2\pi)^2} e^{ip_\perp \cdot (\vec{r}_\perp - \vec{r}'_\perp)} \tilde{G}(\omega, \vec{p})$$
The Fourier transform of the translation invariant part reads
\[
\tilde{G}(\omega, \vec{p}) = ie^{-\vec{p}_\perp l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega, \vec{p})}{\omega^2 - E_n^2}
\]
where
\[
D_n(\omega, \vec{p}) = 2(\omega \gamma^0 - p_z \gamma^3 + m) \left[ \mathcal{P}_- L_n(2\vec{p}_\perp^2 l^2) - \mathcal{P}_+ L_{n-1}(2\vec{p}_\perp^2 l^2) \right] + 4(\vec{p}_\perp \cdot \vec{v}_\perp) L_{n-1}^1(2\vec{p}_\perp^2 l^2)
\]
and the following notation for the spin projectors is used
\[
\mathcal{P}_\pm = \frac{1 \pm i \gamma^1 \gamma^2}{2}
\]
Similarly, in momentum-coordinate space representation:
\[
\tilde{G}(\omega, p_z; \vec{r}_\perp) = i \frac{e^{-\vec{r}_\perp^2/(4l^2)}}{2\pi l^2} \sum_{n=0}^{\infty} \frac{F_n(\omega, p_z; \vec{r}_\perp)}{\omega^2 - E_n^2}
\]
where
\[
F_n(\omega, p_z; \vec{r}_\perp) = 2(\omega \gamma^0 - p_z \gamma^3 + m) \left[ \mathcal{P}_- L_n \left( \frac{\vec{r}_\perp^2 l^2}{2l^2} \right) - \mathcal{P}_+ L_{n-1} \left( \frac{\vec{r}_\perp^2 l^2}{2l^2} \right) \right] - \frac{i}{l^2} (\vec{r}_\perp \cdot \vec{v}_\perp) L_{n-1}^1 \left( \frac{\vec{r}_\perp^2}{2l^2} \right)
\]
**Dimensional Reduction**

- The low-energy dynamics is determined by the lowest Landau level \((n=0)\)
  
  \[ E_0^\pm = \pm p_z \]

- This is a \((1+1)D\) spectrum!

- Propagator is also \((1+1)D\):

  \[
  \tilde{G}_{LLL}(\omega, \vec{p}) = 2ie^{-\vec{p}_\perp^2l^2} \frac{\omega \gamma^0 - p_z \gamma^3}{\omega^2 - p_z^2} \frac{1 - i\gamma^1 \gamma^2}{2}
  \]

- In addition, there is a nonzero density of states at \(E=0\):

  \[
  \left. \frac{dn}{dE} \right|_{E=0} = \frac{1}{\delta E} \left( \frac{N_{\text{max}}}{L_x L_y} \right) \left( \int_0^{\delta E} \frac{dp_z}{2\pi} \right) = \frac{|eB|}{4\pi^2}
  \]
**PAIRING INSTABILITY**

- Thought experiment:
  - Create a particle-antiparticle pair (energy price: $\Delta E$)
  - The pair can form a bosonic bound state (energy gain: $-\epsilon_b$)
  - If $\epsilon_b > \Delta E$, copious formation of bound states is beneficial
  - Note, $\Delta E$ can be arbitrarily small when $m = 0$ (!)
  - The bound states of fermions are bosons
  - Bosons can (and will) occupy the lowest energy state ($\vec{P} = 0$), and thus form a Bose condensate $\langle \bar{\psi} \psi \rangle \neq 0$
  - Ground state (vacuum) changes its properties (e.g., chiral symmetry breaks down, an energy gap opens in spectrum)
**DO BOUND STATES ALWAYS FORM IN 3D?**

- Consider a 3D potential well in quantum mechanics [Landau-Lifshitz, Quantum Mechanics]

\[
U(r) = \begin{cases} 
- g \frac{\pi^2 \hbar^2}{8m^*a^2} & \text{for } r \leq a \\
0 & \text{for } r > a 
\end{cases}
\]

- Bound states form only when the well is deep enough (namely, \(g > 1\)):

\[
|E_{3D}| \approx \frac{\pi^4 \hbar^2}{2^7 a^2 m^*} (g - 1)^2, \quad \text{assuming } 0 < g - 1 \ll 1
\]

- There are no bound states when \(g < 1\), i.e., when the well is not deep enough (in other words, when the coupling constant is not strong enough)
**COMPARE: BOUND STATES IN 1D**

- Bound states always form

\[ |E_{1D}| \approx \frac{m^*_2}{2\hbar^2} \left( -\int_{-\infty}^{+\infty} U(x) \, dx \right)^2 \]

- This is a perturbative result (!)

\[ |E_{1D}| \propto g^2, \quad \text{when} \quad U(x) \to gU(x) \]

- Rigorous statement: at least one bound state exists if

\[ \int (1 + |x|)|U(x)| \, dx < \infty \quad \& \quad \int U(x) \, dx \leq 0 \]

[B. Simon, Annals Phys. 97 (1976) 279]
How about bound states in 2D?

- Bound states always form
  \[ |E_{2D}| \approx \frac{\hbar^2}{a^2 m_*} \exp \left( - \frac{\hbar^2}{m_*} \int_0^\infty r U(r) dr \right)^{-1} \]

- This is a nonperturbative result
  \[ |E_{2D}| \propto \exp \left( - \frac{C}{g} \right), \quad \text{when} \quad U(x) \to g U(x) \]

- Rigorous statement: at least one bound state exists if
  \[ \int |U(x)|^{1+\epsilon} d^2x < \infty, \quad \int (1 + x^2)^\epsilon |U(x)| d^2x < \infty \quad \& \quad \int U(x) d^2x \leq 0 \]

[B. Simon, Annals Phys. 97 (1976) 279]
Universal Magnetic Catalysis

- Quantum field theory of charged fermions ($m=0$) at $\vec{B} \neq 0$
  - Dimensional reduction (caused by a nonzero $\vec{B}$)
  - Nonzero density of states ($\propto |eB|$) at $E=0$
  - Attraction between particles and antiparticles

- Universal outcome:
  - Spontaneous rearrangement of the ground state
  - Breakdown of chiral symmetry
  - Opening a nonzero gap in the Dirac spectrum

  [Gusynin, Miransky, Shovkovy, Phys. Rev. Lett. 73, 3499 (1994)]
  [Shovkovy, Lect. Notes Phys. 871, 13 (2013)]

- This is similar to superconductivity in metals due to Cooper pairing of electrons
Symmetry breaking: Method I

- The Schwinger-Dyson equation for the fermion self-energy/propagator

\[
G^{-1}(x, y) = G_0^{-1}(x, y) + 4\pi\alpha_s \gamma^\mu T^A G(x, y) \gamma^\nu T^B D_{\mu\nu}^{AB}(y - x)
\]

- In coordinate space, e.g.,

\[
G^{-1}(x, y) = G_0^{-1}(x, y) + 4\pi\alpha_s \gamma^\mu T^A G(x, y) \gamma^\nu T^B D_{\mu\nu}^{AB}(y - x)
\]

- Note: both the propagator \(G(x, y)\) and its inverse \(G^{-1}(x, y)\) have the same Schwinger phase \(e^{i\Phi(\vec{r}_\perp, \vec{r}'_\perp)}\)

- Like in a metal, a large density of states at \(E=0\) implies that screening effects are important
Symmetry breaking: Method II

- Homogeneous Bethe-Salpeter equation for a \textit{massless} bound states with quantum numbers of the NG bosons

The NG wave function is defined by

$$\chi_{AB}^\beta(u, u'; P) = -i \int d^4u_1 d^4u'_1 d^4u_2 d^4u'_2 G_{A1}(u, u_1) K_{A_1B_1; A_2B_2}(u_1u'_1, u_2u'_2) \chi_{A_2B_2}^\beta(u_2, u'_2; P) G_{B_1B}(u'_2, u')$$

The NG wave function is defined by \( \chi_{AB}^\beta = \langle 0 | T\psi_A(u) \bar{\psi}_B(u') | P; \beta \rangle \)

and (the ladder approximation) kernel in QED is

$$K_{A_1B_1; A_2B_2}(u_1u'_1, u_2, u'_2) = -4\pi i \alpha \delta_{a_1a_2} \delta_{b_2b_1} \gamma_{\mu}^{\nu} \gamma_{\nu}^{\mu} D_{\mu\nu}(u'_2 - u_2) \delta(u_1 - u_2) \delta(u'_1 - u'_2)$$

+ $$4\pi i \alpha \delta_{a_1b_1} \delta_{b_2a_2} \gamma_{\mu}^{\nu} \gamma_{\nu}^{\mu} D_{\mu\nu}(u_1 - u'_2) \delta(u_1 - u'_1) \delta(u_1 - u'_2)$$
QED IN STRONG MAGNETIC FIELD

- The NG-boson wave function ($r_\mu = u_\mu - u'_\mu$):
  \[
  \chi^\beta_{AB}(u, u'; P) = \lambda^\beta_{ab} e^{-iPR} \exp \left[ -i\epsilon^\mu A^\text{ext}_\mu (R) \right] \tilde{\chi}_{nm}(R, r; P)
  \]

- In the LLL approximation,
  \[
  \varphi(p_\parallel) = \frac{\pi \alpha}{(2\pi)^4} \int d^2 k_\parallel \left( 1 - i\gamma^1 \gamma^2 \right) \gamma^\mu \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel - m_{\text{dyn}}} \varphi(k_\parallel) \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel - m_{\text{dyn}}} \gamma^\nu \left( 1 - i\gamma^1 \gamma^2 \right) D_{\mu\nu}^{\parallel}(k_\parallel - p_\parallel)
  \]
  where we introduced
  \[
  (\hat{p}_\parallel - m_{\text{dyn}}) \tilde{\chi}(p)(\hat{p}_\parallel - m_{\text{dyn}}) = \exp(-l^2 p_\perp^2) \varphi(p_\parallel)
  \]
  and
  \[
  D_{\mu\nu}^{\parallel}(k_\parallel - p_\parallel) = i\pi \delta_{\mu\nu} \int_0^\infty dx \exp(-l^2 x/2) \frac{(k_\parallel - p_\parallel)^2 + x}{(k_\parallel - p_\parallel)^2 + x}
  \]

- The LLL solution has the Dirac structure
  \[
  \varphi(p_\parallel) = A\gamma_5 (1 - i\gamma^1 \gamma^2)
  \]

- The equation for $A(p_\parallel)$ reads
  \[
  A(p_\parallel) = \frac{\alpha}{2\pi^2} \int \frac{A(k_\parallel) d^2 k_\parallel}{k_\parallel^2 + m_{\text{dyn}}^2} \int_0^\infty dx \frac{e^{-x l^2/2}}{x + (k_\parallel - p_\parallel)^2}
  \]
**Effective Schrodinger Problem**

- Rewrite the problem in terms of
  \[ \Psi(\mathbf{r}) = \int \frac{d^2k_\parallel}{(2\pi)^2} \frac{e^{i\mathbf{r} \cdot \mathbf{k}_\parallel}}{k_\parallel^2 + m_{dyn}^2} A(k_\parallel) \]

- Function \( \Psi(\mathbf{r}) \) satisfies the following 2D Schrodinger equation:
  \[ \left[ -\nabla_\mathbf{r}^2 + m_{dy}^2 + V(\mathbf{r}) \right] \Psi(\mathbf{r}) = 0 \]
  where the effective potential is long-ranged
  \[ V(\mathbf{r}) = -\frac{\alpha}{2\pi^2} \int d^2p e^{i\mathbf{p} \cdot \mathbf{r}} \int_0^\infty dx \exp(-x/2) \frac{1}{p^2 + x} \approx -\frac{2\alpha}{\pi} \frac{1}{r^2}, \quad r \to \infty \]

- The lowest energy bound state gives
  \[ m_{dyn} \approx C \sqrt{|eB|} \exp \left[ -\frac{\pi}{2} \left( \frac{\pi}{2\alpha} \right)^{1/2} \right] \] (LLL & weak coupling)
**SCREENING EFFECTS**

- Photon exchange interaction is screened in a strong B-field

\[
\Pi_{\mu\nu} \equiv \gamma_\mu \gamma_\nu \approx (q_\mu q_\nu - q_\parallel^2 g_{\mu\nu}) e^{-q_\perp^2 \lambda^2} \Pi(q_\parallel^2)
\]

- The screened photon propagator reads

\[
\mathcal{D}_{\mu\nu}(q) = -i \left[ \frac{1}{q^2} g_{\mu\nu} + \frac{q_\parallel q_\nu}{q^2 q_\parallel^2} + \frac{1}{q^2 + q_\parallel^2 \Pi(q_\perp^2, q_\parallel^2)} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q_\parallel^2} \right) - \frac{\lambda}{q^2} \frac{q_\mu q_\nu}{q_\parallel^2} \right]
\]

where the polarization function has the asymptotes

\[
\Pi(q_\parallel^2) \sim \frac{\bar{\alpha}}{3\pi} \frac{|eB|}{m^2_{\text{dyn}}}, \quad \text{as } |q_\parallel^2| \ll m^2_{\text{dyn}} \quad \text{(extremely narrow range in } q_\parallel^2)
\]

\[
\Pi(q_\parallel^2) \sim -\frac{2\bar{\alpha}}{\pi} \frac{|eB|}{q_\parallel^2} \quad \text{as } |q_\parallel^2| \gg m^2_{\text{dyn}} \implies \frac{1}{q^2 + q_\parallel^2 \Pi(q_\perp^2, q_\parallel^2)} \approx \frac{1}{q^2 - M_{\gamma}^2}
\]

where the effective photon screening mass is

\[
M_{\gamma}^2 = \frac{2\bar{\alpha}}{\pi} |eB|
\]
With screening effects included,

\[ \chi^\beta \begin{array}{c} \rightarrow \\ \end{array} \chi^\beta \]

The result is similar up to the replacement \( \alpha \rightarrow \alpha/2 \)

\[ m_{\text{dyn}} \approx C \sqrt{|eB|} \exp \left[ -\frac{\pi}{2} \left( \frac{\pi}{\alpha} \right)^{1/2} \right] \]

\( \alpha \rightarrow \alpha/2 \) changes the result dramatically

The improved ladder approximation is not reliable either (!)

- The vertex corrections are important too
- More singularities \( \sim \ln\left( |eB|/m_{\text{dyn}}^2 \right) \sim 1/\sqrt{\alpha} \) at higher orders

Re-summation of infinitely many diagrams is needed (!)
Toward exact result

- QED in a strong B-field looks like \((1+1)D\) Schwinger model
- Use same strategy: fix a gauge in which all (singular) vertex corrections vanish!
- Such a (non-local) gauge exists

\[
D_{\mu\nu}(q) = -i \frac{1}{q^2} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) - id(q^2, q^2) \frac{q_\mu q_\nu}{q^2 q^2}
\]

where

\[
d = -q^2 \Pi/[q^2 + q^2 \Pi] + q^2 / q^2
\]

- Then, the full photon propagator

\[
D_{\mu\nu}(q) = -i \frac{g_{\mu\nu}^\parallel}{q^2 + q^2 \Pi(q^2, q^2)} - i \frac{g_{\mu\nu}^\perp}{q^2} + i \frac{q_\mu q_\nu + q_\mu q_\nu^\parallel + q_\mu q_\nu^\perp}{(q^2)^2}
\]

- No dangerous infrared singularities appear because

\[
P^- \gamma_\mu P^- = \gamma^\parallel,\mu \quad \text{and} \quad \gamma^\parallel,\alpha \gamma^\parallel,\mu_1 \gamma^\parallel,\mu_2 \ldots \gamma^\parallel,\mu_{2n+1} \gamma^\alpha = 0
\]
In such a nonlocal gauge, the dynamical mass is reliable

\[ m_{\text{dyn}} \approx \sqrt{2|eB| (\alpha N_f)^{1/3}} \exp \left[ -\frac{\pi}{\alpha \ln \frac{c_1}{\alpha N_f}} \right], \quad c_1 \approx 1.82 \pm 0.06 \]
The Schwinger-Dyson equation

\[ G^{-1}(x, y) = G_0^{-1}(x, y) + 4\pi\alpha_s \gamma^\mu T^A G(x, y) \gamma^\nu T^B D^{AB}_{\mu\nu}(y - x) \]

- Non-Abelian structure \((T^A T^A = C_2)\): \(\alpha \to \frac{N_c^2 - 1}{2N_c} \alpha_s\)

- Screening effects in the strong field limit \(\sqrt{|eB|} \gg \Lambda_{QCD}\)

\[ \mathcal{P}^{AB,\mu\nu} \sim \frac{\alpha_s}{6\pi} \delta^{AB} \left( k^\mu k^\nu - k^2 g^{\mu\nu} \right) \sum_{q=1}^{N_f} \frac{|e_q B|}{m_q^2}, \quad \text{for } |k^2| \ll m_q^2 \]

\[ \mathcal{P}^{AB,\mu\nu} \sim -\frac{\alpha_s}{\pi} \delta^{AB} \left( k^\mu k^\nu - k^2 g^{\mu\nu} \right) \sum_{q=1}^{N_f} \frac{|e_q B|}{k^2}, \quad \text{for } m_q^2 \ll |k^2| \ll |eB| \]
**Expression for Dynamical Mass**

- The gluon effective mass reads \( m_{dyn}^2 \ll |k_\parallel^2| \ll |eB| \)

\[
M_g^2 \simeq \frac{\alpha_s}{\pi} \sum_f |e_f B| = \frac{\alpha_s}{3\pi} (2N_u + N_d) |eB|
\]

- As in QED, the non-local gauge prevents singular infrared corrections in higher-order diagrams

- Compared to QED: \( \alpha \to \frac{N_c^2 - 1}{2N_c} \alpha_s \) and \( M_\gamma^2 \to M_g^2 \)

- Then,

\[
m_q^2 \simeq 2C_1|e_qB| \left( c_q \alpha_s \right)^{2/3} \exp \left[ -\frac{4N_c\pi}{\alpha_s(N_c^2 - 1) \ln(C_2/c_q\alpha_s)} \right]
\]

where \( C_1 \simeq C_2 \approx 1 \) and \( c_q \approx (2N_u + N_d)|e|/(6\pi|e_q|) \)
QUARK MASS VS. $B$

- Quantitative dependence on the field ($\sqrt{eB} \gg \Lambda_{QCD}$)

![Graph showing the quantitative dependence of quark masses on the field strength](image)

Chiral condensate in lattice QCD

\[ \Sigma = \sum_{\pi} \phi_{\pi} \phi_{\pi} \propto m_{\text{dyn},\pi} \]

\[ \Delta(\Sigma_u + \Sigma_d)/2 \]

\[ \Sigma_f = \bar{\psi}_f \psi_f \propto m_{\text{dyn},f} \]

\[ T=0 \]

Catalysis vs. Inverse Catalysis


Physics Opportunities at a Lepton Collider in the Fully Nonperturbative QED Regime
Inverse Catalysis ($T_C$ vs. $B$)

- Strange quark number susceptibility
- Average light quark condensate
- Polyakov loop

• Gluon screening (?)
• Polyakov loops (?)

[Bruckmann, G. Endrodi, T. G. Kovacs, JHEP 04 (2013) 112]
SUPER-STRONG $B$: PREDICTION

\[ T_c \text{ (MeV)} \]

\[ eB \text{ (GeV}^2) \]

decohminement transition line

crossover
critical endpoint

prediction

first order

Predicted Phase Diagram

$T_c \sim m_{\text{dyn}}$ (chiral symmetry restoration)

$T_c^* \sim \lambda_{\text{QCD}}$ (deconfinement)

Summary

• Strong magnetic field effects:
  – dimensional reduction
  – nonzero density of states at $E=0$
  – enhanced particle-antiparticle pairing dynamics

• Even weakly coupled regime is nonperturbative

• Magnetic catalysis in QED is hidden behind the “large” electron mass

• Strong magnetic field effects are testable in QED-like Dirac materials