Magnetic catalysis in QCD in a superstrong magnetic field

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Magnetic Catalysis: Plan of Lectures

• Dirac fermions in magnetic field
• Dimensional reduction
• Magnetic catalysis: basics
• Magnetic catalysis in toy model
• Magnetic catalysis in QED
• Magnetic catalysis in QCD
• Anisotropic confinement
• Inverse catalysis
• Phase diagram
QCD in Magnetic Fields

- Relativistic collisions of heavy ions produce quark-gluon plasma & strong magnetic fields

$10^{18} - 10^{19}$ Gauss ($\sqrt{|eB|} \sim 100$ MeV)

- Quark matter may form inside magnetars

$10^{14} - 10^{16}$ Gauss ($\sqrt{|eB|} \sim 1$ MeV to 10 MeV)

- Strong magnetic field is an instructive theoretical tool to study confined gauge theories such as QCD

$\geq 10^{19}$ Gauss ($\sqrt{|eB|} \geq 100$ MeV to 10 MeV)
**DIRAC VACUUM**

- At $m = 0$, the Dirac vacuum is a **semimetal**
  - No energy gap between the filled Dirac sea states and the empty positive-energy states ($E = \pm p$)
  - However, the density of states *vanishes* at $E=0$
  - A nonzero electric current could be produced by an arbitrarily small electric field

- At $m \neq 0$, the Dirac vacuum is an **insulator**
  - Energy gap $\Delta E = 2m$ between the antiparticle and particle states ($E = \pm \sqrt{p^2 + m^2}$)
  - The density of states @ $E=0$ *vanishes* (no states)
  - Electric current is exponentially small, i.e., $e^{-\pi m^2/|eE|}$ (due to Schwinger pair creation)
• Lagrangian density for charged Dirac fermions (units with $c = 1$):

$$\mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - m \right) \psi$$

where $D_\mu = \partial_\mu + ieA_\mu$, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ and $g^{\mu\nu} = (1, -1, -1, -1)$

• Consider the following two types of global transformations:

$$\psi \rightarrow e^{i\alpha} \psi \quad \text{and} \quad \psi \rightarrow e^{i\alpha \gamma^5} \psi$$

where $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$

The corresponding Noether’s currents are

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad \text{and} \quad j_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$$

They satisfy the relations:

$$\partial_\mu j^\mu = 0 \quad \text{and} \quad \partial_\mu j_5^\mu = 2i \, m \, \bar{\psi} \gamma^5 \psi$$

Both transformations are symmetries when $m = 0$, but chiral symmetry is broken when $m \neq 0$. [The chiral anomaly may complicate the situation]
Dirac fermions at $B \neq 0$

- Dirac equation for charged fermions:
  \[(i\gamma^\mu D_\mu - m)\psi = 0\]
  where $A_\mu = (A_0, -\vec{A})$ and the Landau gauge $\vec{A} = (-By, 0,0)$ is used.

- Look for a solution in the form: $\psi = (i\gamma^\mu D_\mu + m)\phi$. Then,
  \[\left[-\partial_0^2 + (\partial_x + ieBy)^2 + \partial_y^2 + \partial_z^2 + i\gamma^1\gamma^2 eB - m^2\right]\phi = 0\]

- Normalized solutions for $\phi$ have the form
  \[\phi_{k,\pm} \propto \frac{1 \pm i\text{sgn}(eB)\gamma^1\gamma^2}{2} \varphi_k(y)e^{-i\omega t + ip_{xx} + ip_{zz}}\]
  where $\varphi_k$ are harmonic oscillator wave functions, i.e.,
  \[\varphi_k \propto H_k(\xi)e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l \text{sgn}(eB)\quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}\]

- The dispersion relation is given by
  \[\omega = E_n^\pm = \pm\sqrt{2n|eB| + p_z^2 + m^2}\]
  where $n = k + \frac{1}{2} + \text{sgn}(eB)s_z$ and $s_z = \pm \frac{1}{2}$ is an eigenvalue of $\frac{i}{2}\gamma^1\gamma^2$
The Landau level energies are independent of \( p_x \)

\[
E_n^\pm = \pm \sqrt{2n|eB| + p_z^2 + m^2}
\]

This means that each level is highly degenerate

Let’s calculate the degeneracy by confining the system in a finite box of size \( L_x \times L_y \) with periodic boundary conditions

The wave function is a plane wave in the \( x \) direction: \( \psi(x) \propto e^{ip_xx} \)

\[
\psi(0) = \psi(L_x) \implies e^{ip_x L_x} = 1 \implies p_x = \frac{2\pi n}{L_x}, \quad n = 1, 2, \ldots, N_{\text{max}}
\]

The value of \( p_x \) sets the center of the Landau orbit in \( y \)-direction:

\[
y_c \approx p_x l^2 \implies p_{x,\text{max}} l^2 \lesssim L_y \implies \frac{2\pi N_{\text{max}}}{L_x} \frac{1}{|eB|} \approx L_y \implies \frac{N_{\text{max}}}{L_x L_y} \approx \frac{|eB|}{2\pi}
\]

The degeneracy is proportional to the field strength and the size (area) of the system in the spatial directions perpendicular to \( \vec{B} \)

\[
N_{\text{max}} \approx \frac{|eB|}{2\pi} L_x L_y
\]
**Landau Energy Spectrum**

- Landau energy levels at $m = 0$
  \[ E_n^{\pm} = \pm \sqrt{2n|eB| + p_z^2} \]
  where $n = k + \frac{1}{2} + \text{sgn}(eB)s_z$
  \[ \text{orbital} \quad \text{spin} \]
- Lowest Landau level is *spin polarized*
  \[ E_0^{\pm} = \pm p_z \quad (k = 0, \quad s_z = -\frac{1}{2}) \]
- Density of states at $E=0$:
  \[ \left. \frac{dn}{dE} \right|_{E=0} = \frac{|eB|}{2\pi} \frac{1}{2\pi} = \frac{|eB|}{4\pi^2} \]
- Higher Landau levels ($n \geq 1$) are twice as degenerate:
  (i) $k = n \quad \& \quad s = -\frac{1}{2}$
  (ii) $k = n - 1 \quad \& \quad s = +\frac{1}{2}$
By definition,
\[
G(r, r') = i \left\langle r \left| (i \gamma^\mu D_\mu - m)^{-1} \right| r' \rightangle
\]
\[
= i (i \gamma^\mu D_\mu + m) \left\langle r \left| [(i \gamma^\mu D_\mu - m)(i \gamma^\nu D_\nu + m)]^{-1} \right| r' \rightangle
\]
\[
= i (i \gamma^\mu D_\mu + m) \left\langle r \left| [-D^\mu D_\mu + i \gamma^1 \gamma^2 eB - m^2]^{-1} \right| r' \rightangle
\]
\[
= i (i \gamma^\mu D_\mu + m) \sum \left\langle r \left| k, p_z, s_z \right\rangle (\omega^2 - E_n^2)^{-1} \langle k, p_z, s_z | r' \right\rangle
\]

Note that the explicit form of the wave functions is the same as before
\[
\psi_{k, p_z, s_z}(r) = \langle r | k, p_z, s_z \rangle \propto H_k(\xi) e^{-\frac{\xi^2}{2}} e^{-i \omega t + ip_z z} U_{s_z}, \text{ where } \xi = \frac{y}{l} + p_x l
\]

The final expression for the propagator has the form
\[
G(\omega, p_z; \vec{r}_\perp, \vec{r}'_\perp) = e^{i \Phi(\vec{r}_\perp, \vec{r}'_\perp)} \tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}'_\perp)
\]
where \(\Phi(\vec{r}_\perp, \vec{r}'_\perp) = -e \int_{\vec{r}_\perp}^{\vec{r}'_\perp} A_\nu d r^\nu\) is the Schwinger phase (!), and
\[
\tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}'_\perp) = \int \frac{d^2 \vec{p}_\perp}{(2\pi)^2} e^{i \vec{p}_\perp \cdot (\vec{r}_\perp - \vec{r}'_\perp)} \tilde{G}(\omega, \vec{p})
\]
The Fourier transform of the translation invariant part reads

\[ \tilde{G}(\omega, \vec{p}) = i e^{-\vec{p}_\perp l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega, \vec{p})}{\omega^2 - E_n^2} \]

where

\[ D_n(\omega, \vec{p}) = 2(\omega \gamma^0 - p_z \gamma^3 + m) [\mathcal{P}_+ L_n(2\vec{p}_\perp l^2) - \mathcal{P}_- L_{n-1}(2\vec{p}_\perp l^2)] + 4(\vec{p}_\perp \cdot \vec{y}_\perp) L^1_{n-1} (2\vec{p}_\perp l^2) \]

and the following notation for the spin projectors is used

\[ \mathcal{P}_\pm = \frac{1 \pm \text{sgn}(eB) \gamma^1 \gamma^2}{2} \]

Similarly, in momentum-coordinate space representation:

\[ \tilde{G}(\omega, p_z; \vec{r}_\perp) = i \frac{e^{-\vec{r}_\perp^2/(4l^2)}}{2\pi l^2} \sum_{n=0}^{\infty} \frac{F_n(\omega, p_z; \vec{r}_\perp)}{\omega^2 - E_n^2} \]

where

\[ F_n(\omega, p_z; \vec{r}_\perp) = 2(\omega \gamma^0 - p_z \gamma^3 + m) \left[ \mathcal{P}_+ L_n \left( \frac{\vec{r}_\perp^2}{2l^2} \right) - \mathcal{P}_- L_{n-1} \left( \frac{\vec{r}_\perp^2}{2l^2} \right) \right] - \frac{i}{l^2} (\vec{r}_\perp \cdot \vec{y}_\perp) L^1_{n-1} \left( \frac{\vec{r}_\perp^2}{2l^2} \right) \]
DIMENSIONAL REDUCTION

- The low-energy dynamics is determined by the lowest Landau level \((n=0)\)
  \[ E_{0}^{\pm} = \pm p_{z} \]
- This is a (1+1)D spectrum!
- Propagator is also (1+1)D:
  \[
  \tilde{G}_{LLL}(\omega, \vec{p}) = 2ie^{-\vec{p}_{\perp}^{2}l^{2}} \frac{\omega\gamma^{0} - p_{z}\gamma^{3}}{\omega^{2} - p_{z}^{2}} \frac{1 + isgn(eB)\gamma^{1}\gamma^{2}}{2}
  \]
- In addition, there is a nonzero density of states at \(E=0\):
  \[
  \left. \frac{dn}{dE} \right|_{E=0} = \frac{1}{\delta E} \left( \frac{N_{\text{max}}}{L_{x}L_{y}} \right) \left( \int_{0}^{\delta E} \frac{dp_{z}}{2\pi} \right) = \frac{|eB|}{4\pi^{2}}
  \]
• Thought experiment:
  – Create a particle-antiparticle pair (energy price: $\Delta E$)
  – The pair can form a bosonic bound state (energy gain: $-\epsilon_b$)
  – If $\epsilon_b > \Delta E$, copious formation of bound states is beneficial
  – Note, $\Delta E$ can be arbitrarily small when $m = 0$ (!!)
  – The bound states of fermions are bosons
  – Bosons can (and will) occupy the lowest energy state ($\vec{P} = 0$), and thus form a Bose condensate $\langle \bar{\psi}\psi \rangle \neq 0$
  – Ground state (vacuum) changes its properties (e.g., chiral symmetry breaks down, an energy gap opens in spectrum)
Consider a 3D potential well in quantum mechanics [Landau-Lifshitz, Quantum Mechanics]

\[ U(r) = \begin{cases} 
-\frac{\pi^2 \hbar^2}{8m* a^2} & \text{for } r \leq a \\
0 & \text{for } r > a 
\end{cases} \]

Bound states form only when the well is deep enough (namely, \( g > 1 \)):

\[ |E_{3D}| \approx \frac{\pi^4 \hbar^2}{2^7 a^2 m_*} (g - 1)^2 \text{, assuming } 0 < g - 1 \ll 1 \]

There are no bound states when \( g < 1 \), i.e., when the well is not deep enough (in other words, when the coupling constant is not strong enough)
**COMPARE: BOUND STATES IN 1D**

- Bound states always form

\[
|E_{1D}| \approx \frac{m^*_0}{2\hbar^2} \left( -\int_{-\infty}^{+\infty} U(x) \, dx \right)^2
\]

- This is a perturbative result (!)

\[
|E_{1D}| \propto g^2, \quad \text{when} \quad U(x) \to gU(x)
\]

- Rigorous statement: at least one bound state exists if

\[
\int \left(1 + |x|\right) |U(x)| \, dx < \infty \quad \& \quad \int U(x) \, dx \leq 0
\]

[B. Simon, Annals Phys. 97 (1976) 279]
**How about Bound States in 2D?**

- Bound states always form
  
  \[ |E_{2D}| \approx \frac{\hbar^2}{a^2 m_*} \exp\left(-\frac{\hbar^2}{m_*} \int_0^\infty rU(r)dr\right)^{-1} \]

- This is a non-perturbative result
  
  \[ |E_{2D}| \propto \exp\left(-\frac{C}{g}\right), \text{ when } U(x) \to gU(x) \]

- Rigorous statement: at least one bound state exists if

  \[ \int |U(x)|^{1+\varepsilon} d^2x < \infty, \quad \int (1 + x^2)^\varepsilon |U(x)| d^2x < \infty \quad \& \quad \int U(x) d^2x \leq 0 \]

[B. Simon, Annals Phys. 97 (1976) 279]
Universal Magnetic Catalysis

- Quantum field theory of charged fermions \((m=0)\) at \(\vec{B} \neq 0\)
  - Dimensional reduction (caused by a nonzero \(\vec{B}\))
  - Nonzero density of states \((\propto |eB|)\) at \(E=0\)
  - Attraction between particles and antiparticles
- Universal outcome:
  - Copious particle-antiparticle pairing at low energies
  - Condensation of boson pairs that destabilizes the trivial Dirac vacuum
  - Spontaneous rearrangement of the ground state
  - Breakdown of chiral symmetry
  - Opening a nonzero gap in the Dirac spectrum

\[\text{Gusynin, Miransky, Shovkovy, Phys. Rev. Lett. 73, 3499 (1994)}\]
\[\text{Miransky & Shovkovy, Physics Reports 576, 1 (2015)}\]

- The mechanism is similar to superconductivity in metals due to Cooper pairing of electrons
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Let us consider a Nambu-Jona-Lasino model \((m = 0)\) with four-fermion contact interaction

\[
\mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu \right) \psi + \frac{G}{2} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2 \right]
\]

After the Hubbard–Stratonovich transformation, this is equivalent to

\[
\mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - \sigma - i \gamma^5 \pi \right) \psi - \frac{\sigma^2 + \pi^2}{2G}
\]

where the following composite fields were introduced

\[
\sigma = -G \bar{\psi} \psi \quad \text{and} \quad \pi = -G \bar{\psi} i \gamma^5 \psi
\]

The effective action for the composite fields reads

\[
\Gamma(\sigma, \pi) = -\frac{1}{2G} \int d^4 x (\sigma^2 + \pi^2) - i \text{Tr} \ln[ i \gamma^\mu D_\mu - \sigma - i \gamma^5 \pi ]
\]
**Symmetry of the Model**

- **$U_L(1)$ symmetry transformations,** $\psi \rightarrow e^{i\alpha_L (1-\gamma^5)/2} \psi$
  \[
  \bar{\psi} \psi \rightarrow \cos \alpha_L \bar{\psi} \psi - \sin \alpha_L \bar{\psi} i \gamma^5 \psi
  \]
  \[
  \bar{\psi} i \gamma^5 \psi \rightarrow \sin \alpha_L \bar{\psi} \psi + \cos \alpha_L \bar{\psi} i \gamma^5 \psi
  \]

- **$U_R(1)$ symmetry transformations,** $\psi \rightarrow e^{i\alpha_R (1+\gamma^5)/2} \psi$
  \[
  \bar{\psi} \psi \rightarrow \cos \alpha_R \bar{\psi} \psi + \sin \alpha_R \bar{\psi} i \gamma^5 \psi
  \]
  \[
  \bar{\psi} i \gamma^5 \psi \rightarrow - \sin \alpha_R \bar{\psi} \psi + \cos \alpha_R \bar{\psi} i \gamma^5 \psi
  \]

- In terms of the composite fields, $U_L(1)/U_R(1)$ transformations:
  \[
  \sigma \rightarrow \cos \alpha_L \sigma - \sin \alpha_L \pi
  \]
  \[
  \pi \rightarrow \sin \alpha_L \pi + \cos \alpha_L \sigma
  \]

(Note that $\rho^2 = \sigma^2 + \pi^2$ remains an invariant.)

- Just like the original action $\int \mathcal{L} \, d^4x$, the effective action $\Gamma(\sigma, \pi)$ should be invariant under the symmetry transformations, i.e.,
  \[
  \Gamma(\sigma, \pi) = \Gamma(\rho) + \frac{1}{2} f_{\mu \nu} \left( \partial_\mu \sigma \partial_\nu \sigma + \partial_\mu \pi \partial_\nu \pi \right) + \cdots
  \]
• Let us consider a homogeneous ground state with a uniform $\sigma$
\[
\sigma = -G \langle \bar{\psi} \psi \rangle \neq 0
\]
(Because of the chiral symmetry, we can always set $\pi = 0$.)
• In this case, $\Gamma(\sigma) = -\int V(\sigma) d^4x$, where the effective action is
\[
V(\sigma) = \frac{\sigma^2}{2G} - \frac{i}{2} \int_0^\infty \frac{ds}{s} \text{tr} \left\langle x \left| e^{-is(D^\mu D_\mu - i\gamma^1 \gamma^2 eB + \sigma^2)} \right| x \right\rangle - (\infty)
\]
• By using the Schwinger result [Phys. Rev. 82, 664 (1951)]
\[
\left\langle x \left| e^{-is(D^\mu D_\mu - i\gamma^1 \gamma^2 eB + \sigma^2)} \right| x \right\rangle = \frac{e^{-is\sigma^2 - i\pi/4}}{8(\pi s)^{3/2}} eBs [\cot eBs + \gamma^1 \gamma^2]
\]
• We derive the effective potential (after $s \to -is$):
\[
V(\sigma) = \frac{\sigma^2}{2G} + \frac{eB}{8\pi^2} \int_1^\infty \frac{ds}{s^2} e^{-s\sigma^2} \coth eBs - (\infty)
\]

Effectiveness potential results

Lowest energy ground state is defined by: \( \frac{dV(\sigma)}{d\sigma} = 0 \) (gap equation)

\[
\sigma_{\text{min}} \approx \frac{eB}{\pi} \exp \left( \frac{\Lambda^2}{|eB|} \right) \exp \left( -\frac{4\pi^2}{|eB|G} \right)
\]
In fact, the gap equation at $B=0$ reads

$$\frac{G\Lambda^2 - 4\pi^2}{G} = \sigma^2 \ln \frac{\Lambda^2}{\sigma^2}$$

It has a nontrivial solution $\sigma_{\text{min}} \neq 0$ only when the coupling strength is sufficiently strong, i.e., $G > G_c = \frac{4\pi^2}{\Lambda^2}$
**Dynamical mass**

- Recall:  \( \mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - \sigma - i \gamma^5 \pi \right) \psi - \frac{\sigma^2 + \pi^2}{2G} \)

- The ground state expectation value \( \langle \sigma \rangle = \sigma_{\text{min}} \) determines the dynamical mass of fermions \( m_{\text{dyn}} \) in the new Dirac vacuum.

- Also, the chiral symmetry is broken in a state with \( \langle \sigma \rangle \neq 0 \)
When a continuous global symmetry breaks down, massless Nambu-Goldstone bosons appear in the particle spectrum:

\[(D_\pi)^{-1} = \frac{\delta^4(x)}{G} + i \text{tr}[G(x,0)i\gamma^5 G(0,x)i\gamma^5]\]

The dispersion relation of NG bosons at \(\vec{p} \to 0\)

\[E_\pi = \sqrt{\nu_{\pi,\perp}^2 \vec{p}_\perp^2 + p_Z^2}\]

where \(\nu_{\pi,\perp} \ll 1\) at weak coupling.

The relation for the \(\sigma\)-boson

\[E_\sigma = \sqrt{M_\sigma^2 + \nu_{\sigma,\perp}^2 \vec{p}_\perp^2 + p_Z^2}\]

where \(M_\sigma = 2\sqrt{3} m_{\text{dyn}}\) & \(\nu_{\sigma,\perp} \ll 1\)
Nonzero Temperature

• Partition function:

\[
Z_{T,\mu} = \text{Tr} \left[ \exp \left( - \frac{H - \mu N}{T} \right) \right] \\
= \int [d\psi d\bar{\psi} d\sigma d\pi] \exp \left( i \int_0^{-i/T} dt \int d^3x \left[ \bar{\psi} (i\gamma^\mu D_\mu - \sigma - i\gamma^5\pi)\psi - \frac{\sigma^2 + \pi^2}{2G} \right] \right)
\]

where the fermion/boson fields satisfy (anti)periodic boundary conditions in imaginary time, e.g., \( \psi(0) = -\psi(-i/T) \)

• Note #1: \( Z_{T,\mu} \) is similar to the generating functional at \( T=0 \)

• Note #2: Hubbard–Stratonovich trick \( \iff \) Gaussian integral

• The effective potential is similar to that at \( T=0 \), but with the energy integration replaced by the Matsubara sum:

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{is\omega^2} \rightarrow iT \sum_{n=-\infty}^{\infty} e^{is(i\omega_n)^2}
\]

where \( \omega \rightarrow i\omega_n = i\pi T(2n + 1) \)
EFFECTS OF NONZERO TEMPERATURE

\[ V_{\beta,\mu}(\rho) = V(\rho) - \frac{N}{2\beta \pi^2 l^2} \int_0^\infty dk_3 \left\{ \ln \left[ 1 + e^{-\beta \left( \sqrt{\rho^2 + k_3^2} - \mu \right)} \right] \right. \\
\left. + 2 \sum_{n=1}^\infty \ln \left[ 1 + e^{-\beta \left( \sqrt{\rho^2 + k_3^2 + 2n/l^2} - \mu \right)} \right] + (\mu \rightarrow -\mu) \right\} \]
Notice that at $T = 0$ the chemical potential $\mu$ has no effect on the effective potential when $\sigma > \mu$ (This is not true at $T \neq 0$)
**Symmetry breaking: Methods used**

- Effective potential for the composite field, e.g., \( \sigma = -G \bar{\psi} \psi \)
  \[
  \frac{dV(\sigma)}{d\sigma} = 0 \quad \text{(gap equation)}
  \]

- In NJL, e.g., \( V_{NJL}(\sigma) = \frac{\sigma^2}{2G} + i \text{tr} \ln[i\gamma^\mu D_\mu - \sigma] \), giving
  \[
  \frac{\sigma}{G} - i \text{tr} \left[ \frac{1}{i\gamma^\mu D_\mu - \sigma} \right] = 0 \quad \Rightarrow \quad \sigma = G \text{tr}[G(x, x)]
  \]

- The same gap equation can be obtained from the Schwinger-Dyson equation for the fermion self-energy/propagator

\[
G^{-1}(x, x') - G_0^{-1}(x, x') = -iG \sum_i \Gamma_i [G(x, x)\Gamma_i - \text{tr}\{G(x, x)\Gamma_i\}] \Gamma_i \delta^4(x - x')
\]

where ansatz \( G^{-1}(x, x') = -i \left( i\gamma^\mu D_\mu - m_{dyn} \right) \delta^4(x - x') \) is used
**Another way: pion as a bound state**

- Homogeneous Bethe-Salpeter equation for a *massless* bound state with quantum numbers of the NG boson

  \[ \chi^\beta = \chi^\beta + \chi^\beta \]

- As we’ll see, in NJL model in the strong-field limit, the pion’s wave function in momentum space should have the structure:

  \[ \chi(p; P \to 0) = A(p_\parallel) e^{-p_\perp^2 l^2} \frac{\omega \gamma^0 - p_z \gamma^3 - m}{\omega^2 - p_z^2 - m^2} \gamma^5 P_+ \frac{\omega \gamma^0 - p_z \gamma^3 - m}{\omega^2 - p_z^2 - m^2} \]

where \( A(p_\parallel) \) with \( p_\parallel = (\omega, p_z) \) satisfies a simple integral equation

\[
A(p_\parallel, E) = \frac{G |eB|}{4\pi^3} \int \frac{A(k_\parallel, E) d^2 k_\parallel, E}{k_\parallel, E^2 + m^2}
\]

(here mass parameter \( m \) is treated as a variational parameter)
Auxiliary Schrodinger Problem

- It is instructive to recast the problem in terms of
  \[ \Psi(r_\parallel) = \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{e^{-i r_\parallel \cdot k_\parallel}}{k_\parallel^2 + m^2} A(k_\parallel) \]

- Function \( \Psi(r_\parallel) \) satisfies the following 2D Schrodinger equation:
  \[ \left[ -\nabla_{r_\parallel}^2 + m^2 + V(r_\parallel) \right] \Psi(r_\parallel) = 0 \]
  where \(-m^2\) plays the role of energy \( \epsilon \), and \( V(r_\parallel) \) is a model-dependent potential (as we will see later)

- In the NJL model, \( V(r_\parallel) \) is proportional to a \( \delta \)-function
  \[ V(r_\parallel) = -\frac{G|eB|}{\pi} \delta^2_\Lambda(r_\parallel) = -\frac{G|eB|}{\pi} \int_0^\Lambda d^2 k_\parallel \frac{d^2 k_\parallel}{(2\pi)^2} e^{-i r_\parallel \cdot k_\parallel} \]

- There exists a bound state solution (\( \epsilon_b < 0 \)) in this Schrodinger problem and, thus, also a real solution for \( m \), i.e.,
  \[ m^2 = -\epsilon_b \simeq \Lambda^2 \exp \left( -\frac{4\pi^2}{|eB|G} \right) \] (LLL & weak coupling)
MAGNETIC CATALYSIS IN QED

• Lagrangian density invariant under $SU_L(N_f) \times SU_R(N_f) \times U(1)$

$$\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \bar{\psi}_f (i \gamma^\mu D_\mu) \psi_f$$

where $D_\mu = \partial_\mu + ie (A_\mu + a_\mu)$ and $F^{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

• The Bethe–Salpeter equation for NG states ($\beta = 1, \ldots, N_f^2 - 1$):

$$\chi^\beta_{AB}(u, u'; P) = -i \int d^4 u_1 d^4 u_1' d^4 u_2 d^4 u_2' G_A(u, u_1) K_{A_1 A_2 B_1; A_2 B_2}(u_1 u_1', u_2 u_2') \chi^\beta_{A_2 B_2}(u_2, u_2'; P) G_B(u_2', u')$$

where the wave function is defined by $\chi^\beta_{AB} = \langle 0 | T \psi_A(u) \bar{\psi}_B(u') | P; \beta \rangle$

Diagrammatically

where the kernel (in the ladder approximation) is

$$K_{A_1 A_2 B_1; A_2 B_2}(u_1 u_1', u_2, u_2') = -4\pi i \alpha \delta_{a_1 a_2} \delta_{b_1 b_2} \gamma_{n_1 n_2}^{\mu} \gamma_{m_2 m_1}^{\nu} D_{\mu \nu}(u_2' - u_2) \delta(u_1 - u_2) \delta(u_1' - u_2')$$

$$+ 4\pi i \alpha \delta_{a_1 b_1} \delta_{a_2 b_2} \gamma_{n_1 m_1}^{\mu} \gamma_{n_2 m_2}^{\nu} D_{\mu \nu}(u_1 - u_2) \delta(u_1 - u_1') \delta(u_2 - u_2')$$
**Solution in Strong Field Limit**

- Structure of the NG-boson wave function \( r_\mu = u_\mu - u'_\mu \):

\[
\chi_{AB}^\beta (u, u'; P) = \lambda_{ab}^\beta e^{-iPR} \exp \left[ -i e r_\mu A_{\mu}^{\text{ext}} (R) \right] \tilde{\chi}_{nm} (R, r; P)
\]

- In the LLL approximation, the equation reduces to

\[
\varphi (p_\parallel) = \frac{\pi \alpha}{(2\pi)^4} \int d^2k_\parallel \left( 1 - i \gamma^1 \gamma^2 \right) \gamma^\mu \frac{\hat{k}_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}^2} \varphi (k_\parallel) \frac{\hat{k}_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}^2} \gamma^\nu \left( 1 - i \gamma^1 \gamma^2 \right) D_{\mu \nu}^{\parallel} (k_\parallel - p_\parallel)
\]

where we introduced \((\hat{p}_\parallel - m_{\text{dyn}}) \tilde{\chi} (p) (\hat{p}_\parallel - m_{\text{dyn}}) = \exp (-l^2 p_\perp^2) \varphi (p_\parallel)\)

and

\[
D_{\mu \nu}^{\parallel} (k_\parallel - p_\parallel) = i \pi \delta_{\mu \nu} \int_0^\infty dx \exp \left( -l^2 x / 2 \right) \frac{dx}{(k_\parallel - p_\parallel)^2 + x}
\]

- The solution should have the following Dirac structure

\[
\varphi (p_\parallel) = A \gamma_5 \left( 1 - i \gamma^1 \gamma^2 \right)
\]

- Finally, the equation for \( A (p_\parallel) \) reads

\[
A (p_\parallel) = \frac{\alpha}{2 \pi^2} \int \frac{A (k_\parallel) d^2k_\parallel}{k_\parallel^2 + m_{\text{dyn}}^2} \int_0^\infty dx \frac{e^{-x l^2 / 2}}{x + (k_\parallel - p_\parallel)^2}
\]

Compare with the NJL model.
**REDUCE TO A SCHRODINGER PROBLEM**

- Rewrite the problem in terms of

  \[ \Psi(r) = \int \frac{d^2k_{\parallel}}{(2\pi)^2} \frac{e^{i\mathbf{r} \cdot \mathbf{k}_{\parallel}}}{k_{\parallel}^2 + m_{dyn}^2} A(k_{\parallel}) \]

- Function \( \Psi(r) \) satisfies the following 2D Schrodinger equation:

  \[ \left[ -\nabla_r^2 + m_{dyn}^2 + V(r) \right] \Psi(r) = 0 \]

  where

  \[ V(r) = -\frac{\alpha}{2\pi^2} \int d^2p e^{i\mathbf{p} \cdot \mathbf{r}} \int_0^\infty dx \frac{\exp\left(-x/2\right)}{l^2 p^2 + x} = \frac{\alpha}{\pi l^2} \exp\left(\frac{r^2}{2l^2}\right) \text{Ei}\left(\frac{r^2}{2l^2}\right) \]

- The potential is long-ranged with the following asymptote

  \[ V(r) \simeq -\frac{2\alpha}{\pi} \frac{1}{r^2}, \quad r \to \infty \]

- The lowest energy bound state gives

  \[ m_{dyn} \simeq C \sqrt{|eB|} \exp\left[-\frac{\pi}{2} \left(\frac{\pi}{2\alpha}\right)^{1/2}\right] \quad (\text{LLL & weak coupling}) \]
Photon exchange interaction is screened in a strong B-field

\[ D_{\mu\nu}^{-1}(u, u') = D_{\mu\nu}^{-1}(u - u') + \Pi_{\mu\nu}(u, u') \]

where \( \Pi_{\mu\nu} \equiv (q_\mu q_\nu - q_\parallel g_{\mu\nu}) e^{-q_\perp l^2} \Pi(q_\parallel^2) \)

Then, the screened photon propagator reads

\[ D_{\mu\nu}(q) = -i \left[ \frac{1}{q^2 g_{\mu\nu}} + \frac{q_\parallel q_\nu}{q^2 q_\parallel^2} + \frac{1}{q^2 + q_\parallel^2 \Pi(q_\perp^2, q_\parallel^2)} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q_\parallel^2} \right) - \frac{\lambda}{q^2} \frac{q_\mu q_\nu}{q_\parallel^2} \right] \]

where the polarization function has the asymptotes

\[ \Pi(q_\parallel^2) \simeq \frac{\tilde{\alpha}}{3\pi} \frac{|eB|}{m_{\text{dyn}}^2}, \quad \text{as } |q_\parallel^2| \ll m_{\text{dyn}}^2 \quad \text{extremely narrow range in } q_\parallel^2 \]

\[ \Pi(q_\parallel^2) \simeq -\frac{2\tilde{\alpha}}{\pi} \frac{|eB|}{q_\parallel^2}, \quad \text{as } |q_\parallel^2| \gg m_{\text{dyn}}^2 \quad \Rightarrow \quad \frac{1}{q^2 + q_\parallel^2 \Pi(q_\perp^2, q_\parallel^2)} \simeq \frac{1}{q^2 - M_\gamma^2} \]

where the effective photon screening mass is

\[ M_\gamma^2 = \frac{2\tilde{\alpha}}{\pi} |eB| \]
**IMPROVED LADDER APPROXIMATION**

- Let us re-analyze the problem with screening

\[
A(p_\parallel) = \frac{\alpha}{4\pi^2} \int \frac{A(k_\parallel) d^2 k_\parallel}{k_\parallel^2 + m^2} \int_0^\infty dx \left( \frac{e^{-x l^2/2}}{x + (k_\parallel - p_\parallel)^2} + \frac{e^{-x l^2/2}}{x + (k_\parallel - p_\parallel)^2 + M^2_\gamma} \right)
\]

- Improved vs. simple ladder approximations: \( \alpha \to \alpha/2 \)

- Note, the dynamical mass is very sensitive to small \( \alpha \) (or \( \alpha/2 \)):

\[
m_{d\text{yn}} \simeq C \sqrt{|eB|} \exp \left[ -\frac{\pi}{2} \left( \frac{\pi}{2\alpha} \right)^{1/2} \right] \quad \text{(ladder approximation)}
\]

and, thus, changes drastically with inclusion of screening

- The bigger problem is that the improved ladder approximation is *not* reliable either
  - The vertex corrections will change the result too
  - Singularities \( \sim \ln(\sqrt{|eB|/m_{d\text{yn}}^2}) \sim 1/\sqrt{\alpha} \) in higher-order diagrams

- Re-summation of infinitely many diagrams is needed (!)
TOWARD EXACT RESULT

• QED in a strong field looks almost like (1+1)D

• Lesson from exactly solvable (1+1)D Schwinger model: find a gauge in which all (singular) vertex corrections vanish!

• Such a (non-local) gauge exists

\[ D_{\mu\nu}(q) = -i \frac{1}{q^2} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) - id(q_\perp^2, q_\parallel^2) \frac{q_\parallel^2 q_\parallel^2}{q^2 q_\parallel^2} \]

where

\[ d = -q_\parallel^2 \Pi / [q^2 + q_\parallel^2 \Pi] + q_\parallel^2 / q^2 \]

• The corresponding full photon propagator reads

\[ \mathcal{D}_{\mu\nu}(q) = -i \frac{g_{\parallel,\mu\nu}}{q^2 + q_\parallel^2 \Pi (q_\perp^2, q_\parallel^2)} - i \frac{g_{\perp,\mu\nu}}{q^2} + i \frac{q_\perp^2 q_\perp^2 + q_\parallel^2 q_\parallel^2 + q_\parallel^2 q_\parallel^2}{(q^2)^2} \]

• All potentially dangerous infrared singularities vanish because

\[ \mathcal{P}^+ \gamma_{\mu} \mathcal{P}^+ = \gamma_{\parallel,\mu} \quad \text{and} \quad \gamma_{\parallel,\alpha} \gamma_{\parallel,\mu_1} \gamma_{\parallel,\mu_2} \ldots \gamma_{\parallel,\mu_{2n+1}} \gamma_{\parallel}^\alpha = 0 \]
**RELIABLE STRONG-B LIMIT IN QED**

- Let us use the method of Schwinger-Dyson equation this time:
  \[
  \tilde{G}(x) = \tilde{G}_0(x) - 4\pi\alpha \int d^4y d^4z e^{-i\Phi(x,y) - i\Phi(y,z)} \tilde{G}_0(x-y)\gamma^\mu \tilde{G}(y-z)\gamma^\nu \tilde{G}(z) D_{\mu\nu}(y-z)
  \]
  where all Schwinger phases were carefully accounted for, and the nonlocal gauge is assumed in the photon propagator
  \[
  D_{\mu\nu}^{-1}(x - y) = D_{\mu\nu}^{-1}(x - y) - 4\pi\alpha \text{ tr} [\gamma_\mu \tilde{G}(x-y)\gamma_\nu \tilde{G}(y-x)]
  \]
- Perform Fourier transform and use LLL approximation,
  \[
  \tilde{G}_0(p_\|) = 2ie^{-\hat{p}_\|^2 l^2} \frac{\hat{p}_\|}{p_\|^2} P_+ \quad \text{and} \quad \tilde{G}(p_\|) = 2ie^{-\hat{p}_\|^2 l^2} \frac{\hat{p}_\| + A(p_\|)}{p_\|^2 - A^2(p_\|)} P_+
  \]
- Derive the following gap equation:
  \[
  A(p_\|) = \frac{\alpha}{2\pi^2} \int \frac{d^2k_\| A(k_\|)}{k_\|^2 + A^2(p_\|)} \int_0^\infty dx \frac{e^{-xl^2/2}}{x + (k_\| - p_\|)^2 + M_\gamma^2 e^{-xl^2/2}}
  \]
- Compare with the gap equations in the (improved) ladder QED, obtained with Bethe-Salpeter method
The numerical result is fitted well by

\[ m_{\text{dyn}} \approx \sqrt{2|eB|} \left(\alpha N_f\right)^{1/3} \exp \left[-\frac{\pi}{\alpha \ln\frac{c_1}{\alpha N_f}}\right], \quad c_1 \approx 1.82 \pm 0.06 \]
Magnetic catalysis in QCD in a superstrong magnetic field

XIV International Workshop on Hadron Physics

Lecture #3

Igor Shovkovy
Arizona State University

18-23 March 2018
Florianópolis, SC, Brazil
QCD in Magnetic Field

- QCD is strongly coupled & nonperturbative
- There are theoretical tools that provide insight
  - High-energy (weak-coupling) expansion
  - Large $N_c$ expansion
  - High temperature limit ($T \gg \Lambda_{QCD}$)
  - High density limit ($\mu \gg \Lambda_{QCD}$)
  - Lattice QCD
- Strong magnetic field $B$ is yet another tool
  - it probes physics at short distances $\ell \sim 1/\sqrt{|eB|}$
Lagrangian density of QCD in an external magnetic field

\[ \mathcal{L} = -\frac{1}{2} F_{\mu\nu}^A F_{\mu\nu}^A + \bar{\psi}_f (i \gamma^\mu D_\mu) \psi_f \]

where \( D_\mu = \partial_\mu + ig A_\mu^A \lambda^A / 2 + ie_f A_\mu^{\text{ext}} \)

\[ F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f^{ABC} A_\mu^B A_\nu^C \]

The global chiral symmetry of the model

\[ SU_L(N_u) \times SU_R(N_u) \times SU_L(N_d) \times SU_R(N_d) \times U_A^{-}(1) \]

Quark masses \( m_u \neq m_d \neq 0 \) break the symmetry down to

\[ SU_V(N_u) \times SU_V(N_d) \]
Coupling constant in QCD runs with the energy scale,

\[
\frac{1}{\alpha_s(\mu)} \simeq b \ln \frac{\mu^2}{\Lambda_{QCD}^2}, \quad \text{where} \quad b = \frac{11 N_c - 2 N_f}{12 \pi}
\]

The question is: What happens in a strong magnetic field?
QCD IN STRONG B-FIELD

• Energy scales in the problem at hand

confined gluodynamics, glueballs

Magnetic catalysis in weakly coupled QCD and strong B-field, strong gluon screening

pure (anisotropic) gluodynamics, all massive quarks decoupled,

\[
\frac{1}{\alpha_s(\mu)} \approx b_0 \ln \frac{\mu^2}{\Lambda_{QCD}^2}
\]

deep-UV region with asymptotic freedom and weak B-field

\[
\frac{1}{\alpha_s(\mu)} \approx b \ln \frac{\mu^2}{\Lambda_{QCD}^2}
\]
RUNNING $\alpha_s$ IN QCD AT STRONG B

- In deep UV region $\alpha_s$ is not affected by B-field

$$\lambda_{QCD} \ll \Lambda_{QCD}$$

'Massive gluon'

$$M_g^2 \approx \frac{\alpha_s}{\pi} \sum_f |e_f B|$$

$$m_{dyn}^2 \ll |k_{||}| \ll |eB|$$

$$\frac{1}{\alpha_s} \approx b \ln \frac{|eB|}{\Lambda_{QCD}^2}$$
The general form of the equation is similar to that in QED

\[ G^{-1}(x, y) = G_0^{-1}(x, y) + 4\pi\alpha_s \gamma^\mu T^A G(x, y) \gamma^\nu T^B D_{\mu\nu}^{AB}(y - x) \]

Note that the inverse propagator \( G^{-1}(x, y) \) has the same (!) Schwinger phase as \( G(x, y) \)

- Non-Abelian structure of the theory \( (T^A T^A = C_2) \): \( \alpha \rightarrow \frac{N_c^2 - 1}{2N_c} \alpha_s \)

- Screening effects are included via the polarization function

\[
P^{AB}_{\mu\nu}(x - y) = 4\pi\alpha_s \text{tr}[\gamma_\mu T^A \tilde{G}(x - y) \gamma_\nu \lambda T^A \tilde{G}(y - x)]
\]

- Similar to QED, in the strong field limit \( (\sqrt{|eB|} \gg \Lambda_{QCD}) \)

\[
P^{AB,\mu\nu} \sim \frac{\alpha_s}{6\pi} \delta^{AB} \left( k_\parallel \gamma^\nu - k_\parallel^2 g^{\mu\nu}_\parallel \right) \sum_{q=1}^{N_f} \frac{|e_q B|}{m_q^2}, \quad \text{for } |k_\parallel^2| \ll m_q^2
\]

\[
P^{AB,\mu\nu} \sim -\frac{\alpha_s}{\pi} \delta^{AB} \left( k_\parallel \gamma^\nu - k_\parallel^2 g^{\mu\nu}_\parallel \right) \sum_{q=1}^{N_f} \frac{|e_q B|}{k^2_\parallel m_q^2}, \quad \text{for } m_q^2 \ll |k_\parallel^2| \ll |eB|
Electric and magnetic screening masses on the lattice are fitted well by [Bonati et al., Phys. Rev. D 95, 074515 (2017)]

\[
\frac{m_E^d}{T} = a_E^d \left[ 1 + c_{1;E}^d \frac{|e|B}{T^2} \tan \left( \frac{c_{2;E}^d}{c_{1;E}^d} \frac{|e|B}{T^2} \right) \right]
\]

(and similar for the magnetic one)
**Expression for Dynamical Mass**

- In the region $m_{dyn}^2 \ll |k_\parallel^2| \ll |eB|$, which is most relevant for the fermion-pairing dynamics, the gluon has a “mass”

\[
M_g^2 \approx \frac{\alpha_s}{\pi} \sum_f |e_f B| = \frac{\alpha_s}{3\pi} (2N_u + N_d) |eB|
\]

- As in QED, in order to tame singular infrared corrections in higher-order diagrams, a special non-local gauge is assumed for the gluon propagator.

- Up to replacements $\alpha \to \frac{N_c^2 - 1}{2N_c} \alpha_s$ and $M_\gamma^2 \to M_g^2$, the gap equation looks as in QED. Thus,

\[
m_q^2 \approx 2C_1|e_q B| (c_q \alpha_s)^{2/3} \exp \left[ -\frac{4N_c \pi}{\alpha_s (N_c^2 - 1) \ln(C_2/c_q \alpha_s)} \right]
\]

where $C_1 \approx C_2 \approx 1$ and $c_q \approx (2N_u + N_d)|e|/(6\pi|e_q|)$
Quantitatively, dynamical masses are \( (\sqrt{|eB|} \gg \Lambda_{QCD}) \)

\[ m_u [N_u=1, N_d=2] \]
\[ m_d [N_u=1, N_d=2] \]
\[ m_u [N_u=2, N_d=2] \]
\[ m_d [N_u=2, N_d=2] \]

\[ \ln(|eB|/\Lambda_{QCD}^2) \]

\[ \alpha_s \approx 0.1 \]

\[ T=0 \]
CHIRAL CONDENSENATE IN LATTICE QCD

\[ \Sigma_f = \bar{{\psi}}_f \psi_f \propto m_{\text{dyn},f} \]

\[ T=0 \]

NAMBU-GOLDSTONE BOSONS (PIONS)

- Original global chiral symmetry
  \[ SU_L(N_u) \times SU_R(N_u) \times SU_L(N_d) \times SU_R(N_d) \times U_A^{(-)}(1) \]
  breaks down to
  \[ SU_V(N_u) \times SU_V(N_d) \]

- A total number of broken-symmetry generators: \( N_u^2 + N_d^2 - 1 \)

- Thus, there should be \( (N_u^2 + N_d^2 - 1) \) massless NG bosons

- The unitary pion fields can be written in terms of the coset space generators
  \[ \Sigma_u \equiv \exp \left( i \sum_{A=1}^{N_u^2-1} \lambda^A \pi_u^A / f_u \right), \Sigma_d \equiv \exp \left( i \sum_{A=1}^{N_d^2-1} \lambda^A \pi_d^A / f_d \right) \]
  and \[ \tilde{\Sigma} \equiv \exp \left( i \sqrt{2} \tilde{\pi} / \tilde{f} \right) \]

- In a very strong magnetic field another light pseudo-NG boson, associated with anomalous \( U_A(1) \), may appear
NAMBU-GOLDSTONE BOSONS (PIONS)

• The low-energy effective action should have the form

\[ \mathcal{L}_{NG} \simeq \frac{f_u^2}{4} \text{tr} \left( g_\parallel^{\mu\nu} \partial_\mu \Sigma_u \partial_\nu \Sigma_u^\dagger + v_u^2 g_\perp^{\mu\nu} \partial_\mu \Sigma_u \partial_\nu \Sigma_u^\dagger \right) + \ldots \]

• The pion decay constants are defined by

\[
i \left\langle 0 \right| \bar{\psi} \gamma^\mu \gamma^5 \frac{\lambda^A}{2} \psi \left| \pi^B (P) \right\rangle = P^\mu f_\pi \delta^{AB} = -i \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left( \gamma^\mu \gamma^5 \frac{\lambda^A}{2} \chi_q (k, P) \right) \]

where \( P^\mu = \left( P^0, v_\perp \vec{P}_\perp, P^3 \right) \)

• The Bethe-Salpeter wave function looks like in QED in LLL approximation. So, we find that \( v_\perp^2 \approx 0 \), and

\[
f_q^2 = 4N_c \int \frac{d^2 k_\perp d^2 k_\parallel}{(2\pi)^4} \exp \left( -\frac{k_\perp^2}{|e_q B|} \right) \frac{m_q^2}{(k_\parallel^2 + m_q^2)^2}
\]

which can be easily calculated, giving

\[
f_u^2 = \frac{N_c |eB|}{6\pi^2} \quad \text{and} \quad f_d^2 = \frac{N_c |eB|}{12\pi^2}
\]
Massive quarks decouple from the low-energy dynamics

Gluons are the only “light” degrees of freedom

Assuming that $\Lambda_{QCD}^2 \ll m_{dyn}^2$, the gluodynamics has a semi-perturbative region, $|k_\parallel^2| \lesssim m_{dyn}^2$, where

$$\frac{1}{\tilde{\alpha}_s(\mu)} - \frac{1}{\alpha_s} \simeq b_0 \ln \frac{\mu^2}{m_{dyn}^2}$$

Here $b_0 = \frac{11 N_c}{12\pi}$ and $\frac{1}{\alpha_s} \simeq b \ln \frac{|eB|}{\Lambda_{QCD}^2}$ (Recall: $b = \frac{11 N_c - 2 N_f}{12\pi}$)

Then, we find that the new confinement scale where $\tilde{\alpha}_s = \infty$:

$$-b \ln \frac{|eB|}{\Lambda_{QCD}^2} \simeq b_0 \ln \frac{\lambda_{QCD}^2}{m_{dyn}^2} \Rightarrow \lambda_{QCD} = m_{dyn} \left( \frac{\Lambda_{QCD}}{\sqrt{|eB|}} \right)^{b/b_0}$$
LOW-ENERGY GLUODYNAMICS

- Quadratic part of low-energy effective action for gluons

\[ \mathcal{L}^{(2)}_{\text{glue,eff}} = -\frac{1}{2} \sum_{A=1}^{N_c^2-1} A^A_\mu (-k) \left[ g^{\mu \nu} k^2 - k^\mu k^\nu + \kappa \left( g^{\mu \nu} k^2 - k^\mu k^\nu \right) \right] A^A_\nu (k) \]

where the susceptibility \( \kappa \) is extracted from the polarization tensor \( \mathcal{P}^{AB}_{\mu \nu} \) in the region \(|k|^2 \ll m^2_{\text{dyn}}\), i.e.,

\[ \kappa = \frac{\alpha_s}{6\pi} \sum_{q=1}^{N_f} \frac{|e_q B|}{m^2_q} = \frac{1}{12C_1\pi} \sum_{q=1}^{N_f} \left( \frac{\alpha_s}{c^2_q} \right)^{1/3} \exp \left( \frac{4N_c\pi}{\alpha_s(N_c^2 - 1) \ln(C_2/c_q\alpha_s)} \right) \gg 1 \]

- The requirement of gauge invariance allows to write down the complete expression for the gluon action

\[ \mathcal{L}_{\text{glue,eff}} \sim \frac{1}{2} \sum_{A=1}^{N_c^2-1} \left( E^A_\perp \cdot E^A_\perp + \epsilon E^A_3 E^A_3 - B^A_\perp \cdot B^A_\perp - B^A_3 B^A_3 \right) \]

where \( \epsilon = 1 + \kappa \) is a chromo-dielectric constant (note \( \epsilon \gg 1 \)), \( E^A_i = F^A_{0i} \) and \( B^A_i = \frac{1}{2} \epsilon_{ijk} F^A_{jk} \) are chromo-fields
By using the guidance from an analogous anisotropic QED, the static potential between a pair of quarks should be given by

\[ V(x, y, z) \approx \frac{g_s^2}{4\pi \sqrt{z^2 + \epsilon(x^2 + y^2)}} \]

which is valid for a range of distance scales \( m_{dyn}^{-1} \approx r \lesssim \lambda_{QCD}^{-1} \)

Note that the effective coupling constants

\[ \alpha_s^\parallel = \frac{g_s^2}{4\pi v_g^\parallel} \approx \frac{g_s^2}{4\pi}, \quad \text{where} \quad v_g^\parallel \approx 1 \]

\[ \alpha_s^\perp = \frac{g_s^2}{4\pi \sqrt{\epsilon} v_g^\perp} \approx \frac{g_s^2}{4\pi}, \quad \text{where} \quad v_g^\perp \approx \frac{1}{\sqrt{\epsilon}} \]

are approximately the same in all directions

A posteriori, this naïve “isotropy” may justifies the use of running behavior as in isotropic gluodynamics (not rigorous)
Quark-antiquark potential was fitted by Cornell potential,

\[ V(r) = -\frac{\alpha}{r} + \sigma r + V_0 \]

where \( \sigma \) is the string tension and \( \alpha \) is the Coulombic coefficient.
The dependence of the potential as a function of angle $\theta$ between $\vec{B}$ and $q\bar{q}$ orientation [Bonati et al., Phys. Rev. D 94, 094007 (2016)]

$$V(r, \theta; B) = -\frac{\alpha(\theta; B)}{r} + \sigma(\theta; B)r + V_0(\theta; B)$$

- With increasing angle $\theta$, the string tension increases
NONZERO TEMPERATURE

• What to expect at nonzero temperature (in strong B limit)?

• Very low temperatures, $T \ll \lambda_{QCD}$
  – Ground state is not affected much
  – Color is confined, lowest energy states are glueballs
  – Chiral symmetry is broken ($T \ll \lambda_{QCD} \ll m_{dyn}$)

• Intermediate temperatures, $\lambda_{QCD} \ll T \ll m_{dyn}$
  – Color is deconfined; gluons are thermally populated
  – Chiral symmetry is still broken ($\lambda_{QCD} \ll T \ll m_{dyn}$)

• Moderately high temperatures, $m_{dyn} \ll T \ll \sqrt{|eB|}$
  – Chiral symmetry is restored ($m_{dyn} \ll T$)
Predicted phase diagram

$T_c \sim m_{\text{dyn}}$ (chiral symmetry restoration)

$T_c^\ast \sim \lambda_{\text{QCD}}$ (deconfinement)

INVERSE CATALYSIS AT $T \neq 0$

**Inverse Catalysis at $T \neq 0$**

- The temperature dependence at several fixed values of $B$

- Confinement strongly affects the low-temperature region

\[ \Sigma_{fit} = \frac{\Sigma_0}{\exp\left(\frac{T - T_c}{\Delta T}\right) + 1} \]
DEPENDENCE OF $T_C$ VS. $B$

**Valence vs. Sea**

![Graphs showing data points for different values of $eB$ (GeV$^2$) and $T$ (MeV) for various $N_t$ values.]

- Gluon screening (?)
- Polyakov loops (?)

Or, perhaps, something else (?)

[Bruckmann, G. Endrodi, T. G. Kovacs, JHEP 04 (2013) 112]
SUPER-STRONG $B$: PREDICTION

PREDICTED PHASE DIAGRAM