One-Loop Low Energy Effective Action in QED
(worldline method)
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References:


- V.P. Gusynin and I.A. Shovkovy, hep-th/9804143
Outline

1. Effective action in QED
   - its meaning and range of validity;
   - status of the problem.

2. Method of calculation
   - worldline formalism;
   - perturbation theory and Feynman rules.

3. General result (derivative expansion).

4. Two special cases.

5. Conclusion.
1. The effective action in QED

The original theory $\rightarrow$ an effective theory
(exact) (approximate)

- Reason:
  to get rid of irrelevan degrees of freedom.

- Low-energy dynamics in QED

  1. massive fermions decouple;
  2. electromagnetic field is relevant;
  3. virtual fermions $\rightarrow$ vacuum currents;
  4. Maxwell equations $\rightarrow$ nonlinear.
• In QED $\int (\text{fermions}) \rightarrow$ Effective Action.

• Results ($F^{\mu\nu}$ is constant or varies slowly).

1. The Heisenberg–Euler eff. action (1936)

\[
\mathcal{L}_{HE} = \mathcal{F} + \frac{2\alpha^2(4\mathcal{F}^2 + 7\mathcal{G}^2)}{45\pi^2m^4} + \ldots \tag{1}
\]

where $\mathcal{F} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\mathcal{G} = \frac{1}{8}e^{\mu\nu\lambda\kappa}F_{\mu\nu}F_{\lambda\kappa}$.

2. The Schwinger effective action (1951)

\[
\mathcal{L}_S = \mathcal{F} + \int_0^{\infty} \frac{ds}{8\pi^2s^3}e^{-im^2s}\left[esK_+ \coth(esK_+) \times esK_+ \cot(esK_-) - 1 - \frac{2e^2s^2}{3}\mathcal{F}\right], \tag{2}
\]

where $K_{\pm} = \sqrt{\mathcal{F}^2 + \mathcal{G}^2 \pm \mathcal{F}}$. 

• Derivative expansion

1. The Hauknes expansion in $1/m^2$ (1984)

\[ \mathcal{L}_H = \mathcal{L}_{HE} + \frac{\alpha}{360\pi m^2} \left( \frac{1}{9} F^{\mu\nu} F_{\mu\nu,\lambda}^\lambda + \frac{7}{2} F^{\mu\nu,\lambda} F_{\mu\nu,\lambda} - F^{\lambda\mu,\nu} F_{\lambda\mu,\nu} \right) + O \left( \frac{\alpha^2}{m^6} \right), \]  

2. Derivative expansion for a special case (Lee, Pac, Shin, 1989) in spinor QED$_{3+1}$:

\[ \mathcal{G} = 0 \quad \text{and} \quad F^{\mu\nu} = \Phi(x) F^{\mu\nu} \]

\[ \mathcal{L}_{der} = \mathcal{L}_S - i \frac{\partial \mu \Phi \partial^\mu \Phi}{(8\pi)^2 e^2 \Phi^4} \int_0^\infty \frac{ds}{s} e^{-im^2s} \]

\[ \times \frac{d^3}{ds^3} \left[ es\Phi \tanh(es\Phi) \right] \]

3. Arbitrary background $F_{\mu\nu} = \text{const}$?

OUR RESULT (below)
2. Method of calculation

- One-loop effective action

\[ W^{(1)}(A) \equiv \int d^nx \mathcal{L}^{(1)} = -i \ln \text{Det}(i\overline{D} - m) \]

\[ = -\frac{i}{2} \ln \text{Det} \left( D^2_{\mu} + \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} + m^2 \right), \tag{5} \]

so that

\[ \mathcal{L}^{(1)}(A) = \frac{i}{2} \int_0^\infty \frac{d\tau}{\tau} e^{-im^2\tau} \text{tr} \langle x | \exp(-i\tau H) | x \rangle, \tag{6} \]

where the “Hamiltonian” is given by

\[ H = -\Pi_\mu \Pi^\mu + \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu}(x), \tag{7} \]

\[ \Pi_\mu = -i(\partial_\mu - ieA_\mu). \]

- \( \text{tr} \langle z | U(\tau) | y \rangle \) — a quantum mechanical (!) evolution operator.
• Quantum mechanical path integral (!):

\[ tr(z|U(\tau)|y) = \frac{1}{M} \int \mathcal{D}[x, \psi] e^{iS_{bos} + iS_{fer}} \]  

with

\[ L_{bos}(x) = -\frac{1}{4} \frac{dx_\nu}{dt} \frac{dx^\nu}{dt} - eA_\nu(x) \frac{dx^\nu}{dt} , \]  

\[ L_{fer}(\psi, x) = i\psi_\nu \frac{d\psi^\nu}{dt} - ie\psi^\nu \psi^\lambda F_{\nu\lambda}(x). \]

and boundary conditions

\[ x(0) = y, \quad x(\tau) = z, \quad \psi(0) = -\psi(\tau). \]  

• Fock-Schwinger gauge

\[ A_\nu(x) = \frac{1}{2} (x^\lambda - y^\lambda) F_{\lambda\nu}(y) \]

\[ + \frac{1}{3} (x^\lambda - y^\lambda)(x^\sigma - y^\sigma) \partial_\sigma F_{\lambda\nu}(y) \]

\[ + \frac{1}{8} (x^\lambda - y^\lambda)(x^\sigma - y^\sigma)(x^\mu - y^\mu) \partial_\sigma \partial_\mu F_{\lambda\nu}(y) + \ldots \]  

• \( F_{\mu\nu} = \text{const} \rightarrow \) Gaussian integral
• Derivative terms → “interactions”

• Well defined perturbative expansion in $N_{\text{der}}$

• Feynman rules

1. All vacuum diagrams

2. The propagators:

\[
\langle Tx_\mu(t_1)x_\nu(t_2) \rangle \quad \rightarrow \quad \mu \quad \ldots \quad \nu
\]

\[
\langle T\psi_\mu(t_1)\psi_\nu(t_2) \rangle \quad \rightarrow \quad \mu \quad \rightarrow \quad \nu
\]

3. The interactions

\[
\frac{eF_{\nu_0\nu_1\cdots\nu_{n+1}}}{n!(n+2)} \frac{dx^{\nu_0}}{dt} x^{\nu_1} \cdots x^{\nu_{n+1}}, \quad \text{(Fig. 1)}
\]

\[
\frac{ie}{n!} F_{\lambda\mu,\nu_1\cdots\nu_n} \psi^\lambda \psi_\mu x^{\nu_1} \cdots x^{\nu_n}, \quad \text{(Fig. 2)}
\]

generate two types of VERTICES
• A few comments on the pertur. theory
  1. Infinite number of different vertices
  2. Still, a finite number of diagrams in each order, $N_{der}$, of the pertur. theory
  3. Disconnected diagrams are included
  4. All derivatives come from the vertices
• For example, (a 2-derivative contribution)
• Problem of propagators,
\[ D_{\mu\nu}(t_1, t_2) = D(t_1, t_2, F.)_{\mu\nu} \] (???)

• Set of matrices \( A_{(j)\mu\nu} \):
\[ F^{\lambda\mu} A_{(j)\mu\nu} = A^{\lambda\mu}_{(j)} F_{\mu\nu} = f_j A_{(j)\nu}^{\lambda} \] (13)
where
\[ f_0 = 0, \quad f_{\pm 1} = \pm \sqrt{2F} \quad (2 + 1)D \] (14)
\[ f_{1,2} = \pm iK_-, \quad f_{3,4} = \pm K_+ \quad (3 + 1)D \] (15)

• \( A^{\mu\nu}_{(j)} \) satisfy the identities
\[ \sum_j A^{\mu\nu}_{(j)} = \eta^{\mu\nu}, \quad A^{\mu}_{(j)\mu} = 1, \quad (16) \]
\[ A^{\mu\nu}_{(k)} A_{(j)\nu\lambda} = A^{\mu}_{(j)\lambda} \delta_{kj} \] (17)

• Therefore,
\[ D(t_1, t_2, F.)_{\mu\nu} = \sum_j D(t_1, t_2, f_j) A_{(j)\mu\nu} \] (18)
3. General result (derivative expansion)

\[
\langle x|U_{bos}(\tau)|x\rangle = \langle x|U_{bos}(\tau)|x\rangle_0 \\
\times \left\{ 1 - \frac{i}{8} e F_{\nu\lambda,\mu\kappa} \sum_{j,l} C^V_{(j,l)} (A^{\nu\lambda}_{(j)} A^{\mu\kappa}_{(l)} + 2A^{\nu\mu}_{(j)} A^{\lambda\kappa}_{(l)} ) \\
+ \frac{i}{18} e^2 F_{\nu\lambda,\mu,\rho,\sigma,\kappa,\lambda} \sum_{j,l,k} \left[ C^{VV}_{1(\nu\lambda,\mu,\rho,\sigma,\kappa,\lambda)} A^{\nu\lambda}_{(j)} A^{\mu\sigma}_{(l)} A^{\lambda\kappa}_{(k)} + C^{VV}_{2(\nu\lambda,\mu,\rho,\sigma,\kappa,\lambda)} A^{\nu\lambda}_{(j)} A^{\mu\sigma}_{(l)} A^{\lambda\kappa}_{(k)} \right. \\
+ C^{VV}_{3(\nu\lambda,\mu,\rho,\sigma,\kappa,\lambda)} A^{\nu\lambda}_{(j)} A^{\mu\sigma}_{(l)} A^{\lambda\kappa}_{(k)} + C^{VV}_{4(\nu\lambda,\mu,\rho,\sigma,\kappa,\lambda)} A^{\nu\lambda}_{(j)} A^{\mu\rho}_{(l)} A^{\lambda\kappa}_{(k)} + C^{VV}_{5(\nu\lambda,\mu,\rho,\sigma,\kappa,\lambda)} A^{\nu\lambda}_{(j)} A^{\mu\rho}_{(l)} A^{\lambda\kappa}_{(k)} \right]\right\}
\]  

(19)

(in scalar QED) where

\[
\langle x|U_{bos}(\tau)|x\rangle_0^{3+1} = -\frac{i}{(4\pi\tau)^2 \sin(e\tau K_-) \sinh(e\tau K_+)} \frac{(e\tau K_-)(e\tau K_+)}{(4\pi\tau)^2 \sin(e\tau K_-) \sinh(e\tau K_+)}
\]

and

\[
\langle x|U_{bos}(\tau)|x\rangle_0^{2+1} = -\frac{\exp[-i\pi/4]}{(4\pi\tau)^{3/2}} \frac{(e\tau \sqrt{2F})}{\sinh(e\tau \sqrt{2F})}
\]

\[
\frac{(e\tau \sqrt{2F})}{\sinh(e\tau \sqrt{2F})}
\]
• Expansion in powers of $1/m^2$ (in $\tau$)

\[
\text{tr}\langle x|U_{\text{scal}}(\tau)|x\rangle = \text{tr}\langle x|U_{\text{scal}}(\tau)|x\rangle_0 + \frac{ie^2\tau^3}{30} F^{\nu\lambda} F^{\nu\lambda,\mu} \mu - \\
- \frac{ie^2\tau^3}{180} \left( 4 F^{\nu\lambda,\mu} F^{\nu\lambda,\mu} + F^{\nu\lambda,\mu} \right) + \ldots \}
\]

in scalar QED, and

\[
\text{tr}\langle x|U_{\text{spin}}(\tau)|x\rangle = \text{tr}\langle x|U_{\text{spin}}(\tau)|x\rangle_0 \quad (21)
\times \left\{ 1 + \frac{ie^2\tau^3}{20} F^{\nu\lambda} F^{\nu\lambda,\mu} \right. + \\
+ \frac{ie^2\tau^3}{180} \left( \frac{7}{2} F^{\nu\lambda,\mu} F^{\nu\lambda,\mu} - F^{\nu\lambda,\mu} \right) + \ldots \}
\]

in spinor QED.

• Range of validity

\[
\frac{|eF|}{m^2} \ll 1 \quad \text{and} \quad \frac{\partial|eF|^2}{m^2(eF)^2} \ll 1
\]

• Eq. (21) $\rightarrow$ Hauknes result.
4. Two special cases

- Pure magnetic background

**Scalar QED**

\[
\mathcal{L}_{\text{der}}^{(D)}(B) = \frac{e^2 (\partial_\perp B)^2}{(4\sqrt{\pi})^D |eB|}\left(\frac{6-D}{2}\right) \int_0^\infty \frac{d\omega}{\omega^{D-2}} e^{-\frac{m^2}{|eB|}\omega} \left(\frac{d^3}{d\omega^3} + \frac{d}{d\omega}\right) \left(\frac{\omega}{\sinh \omega}\right),
\]

--- Spatial derivatives → energy decrease (if $|eB|/m^2 < \text{Const}$).

**Spinor QED**

\[
\mathcal{L}_{\text{der}}^{(D)}(B) = -\frac{e^2 (\partial_\perp B)^2}{4(2\sqrt{\pi})^D |eB|}\left(\frac{6-D}{2}\right) \int_0^\infty \frac{d\omega}{\omega^{D-2}} e^{-\frac{m^2}{|eB|}\omega} \left(\frac{d^3}{d\omega^3}\right)(\omega \coth(\omega)),
\]

--- Spatial derivatives → energy decrease.

- An instability?
• Pure electric background

**Spinor QED**

\[
\mathcal{L}_{der}^{(D)}(E) = \frac{e^2 (\partial_{\perp} E)^2}{4(2\sqrt{\pi})^D |eE|^{6-D/2}} \int_0^\infty \frac{d\omega}{\omega^{D-2}} e^{-i\frac{m^2}{|eE|}\omega} \times \frac{d^3}{d\omega^3} (\omega \coth \omega). \quad (24)
\]

where \((\partial_{\perp} E)^2 \equiv (\partial_0 E \partial_0 E - \partial_1 E \partial_1 E)\).

• Correction to \(\text{Im} \mathcal{L}(E)\)
(by definition, \(\sigma \equiv \frac{m^2 \pi}{|eE|}\))

\[
\text{Im} \mathcal{L}_{der}^{(2+1)} = \frac{e^2 (\partial_{\perp} E)^2}{2^8 \pi^3 |eE|^{3/2}} \sum_{n=1}^\infty \frac{\exp(-\sigma n)}{n^{5/2}} \times [15 + 18\sigma n + 12(\sigma n)^2 + 8(\sigma n)^3], \quad (25)
\]

\[
\text{Im} \mathcal{L}_{der}^{(3+1)} = \frac{e^2 (\partial_{\perp} E)^2}{2^6 \pi^4 |eE|} \sum_{n=1}^\infty \frac{\exp(-\sigma n)}{n^3} \times [6 + 6\sigma n + 3(\sigma n)^2 + (\sigma n)^3], \quad (26)
\]
• The corrections are finite as \( m \to 0 \)
5. Conclusion

1. Derivative expansion of the one-loop low-energy effective action in QED$_{2+1}$ and QED$_{3+1}$ (scalar and spinor).

2. All known limits.

3. Explicit expressions for the probability of the particle pairs creation by an electric field due to derivatives.

4. A direct test of breaking the derivative expansion as $m \to 0$.

5. Looking for applications...