Spin, Vortices, Anomaly and Hydrodynamics

Theoretical Physics Colloquium
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Spin

- Most quantum object in Nature: Spin $\frac{1}{2}$ has two basis $\{ | \uparrow \rangle, | \downarrow \rangle \}$

Stern-Gelach Experiment with Ag atoms (1922)

By Tatoute - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=34095239
Relativistic particles: Helicity

\[ \vec{S} \uparrow \bullet \downarrow \vec{p} \]

- Relativistic massless particles: \( \vec{S} = h \frac{\vec{p}}{|\vec{p}|} \), where \( h \) is called helicity

- Under the parity \( \vec{x} \to -\vec{x} \) transformation (P), \( \vec{S} \to -\vec{S} \) and \( \vec{p} \to -\vec{p} \), and helicity flips sign under P

- Any observable that correlates \( \vec{S} \) and \( \vec{p} \) breaks Parity symmetry!
Parity breaking in Electro-Weak Theory

- Lee-Yang's proposal (1956) and the Wu's experiment (1956)

\[ ^{60}\text{Co} \rightarrow ^{60}\text{Ni} + e + \bar{\nu} \]

\[ \langle \vec{S} \cdot \vec{p}_e \rangle < 0 \]

\[ \vec{S} \rightarrow 60\text{Co} \rightarrow 60\text{Ni} + e + \bar{\nu} \]
Λ-polarization in RHIC

Λ (uds)-baryon self-analyzes its spin direction (1950)
\[
dW \frac{1}{d\Omega} = \frac{1}{4\pi}(1 + \alpha \cos \theta), \quad \alpha = 0.642
\]

Life-time \( \tau \sim 10^{-10} \text{s} \) with weak decay to \( p + \pi^- \)

N.B. \( \alpha = -0.642 \) for \( \bar{\Lambda} \) due to CP-conservation

STAR measurement of global Λ polarization. Figure from Nature 548, 62-65(2017) (STAR)
Proton Spin Puzzle

\[ \vec{S}_{\text{total}} = \frac{\hbar}{2} \]

\[ \frac{\hbar}{2} = L_{\text{quark}} + L_{\text{gluon}} \]

\[ L_{\text{quark}} < L_{\text{gluon}} \]

Q: Where does the angular momentum go?

Topological Fluctuations flipping Quark Helicity

(Tarasov-Venugopalan)
Chiral Anomaly

Chiral Anomaly:

\[
\Psi_L \quad \Psi_R
\]

PARTICLE

\(q_L\)

\(h=-\frac{1}{2}\)

\(Q=+1\)

\(q_R\)

\(h=\frac{1}{2}\)

\(Q=+1\)

\(\bar{q}_R\)

\(h=\frac{1}{2}\)

\(Q=-1\)

\(\bar{q}_L\)

\(h=-\frac{1}{2}\)

\(Q=-1\)

Axial charge \(n_A\) is the net helicity density

\[n_A = N(q_L) + N(\bar{q}_L) - (N(q_R) + N(\bar{q}_R))\]

It is P-odd and CP-odd (C-even)

Chiral Anomaly:

\[
\frac{dn_A}{dt} = \frac{e^2}{2\pi^2} \vec{E} \cdot \vec{B} \quad (P-,CP-odd)
\]

Triangle Diagram

Axial charge = (Blue) - (Red)
Chiral Magnetic Effect

(Fukushima-Kharzeev-Warringa, Vilenkin, Son-Zhitnitsky)

Spin alignment in magnetic field leads to momentum alignment to induce a net charge current

\[ \mathbf{J} = \frac{e^2}{2\pi^2 \mu_A} \mathbf{B} \]

and

\[ \mathbf{J}_A = \frac{e^2}{2\pi^2} \mu \mathbf{B} \]

Power \( P = \mathbf{E} \cdot \mathbf{J} = \frac{dn_A}{dt} \mu_A = \frac{e^2}{2\pi^2} \mathbf{E} \cdot \mathbf{B} \mu_A \)

(Nielsen-Ninomiya)
Iso-baric Ru-Zr at RHIC

Taken from Kharzeev-Liao, Nature Reviews Physics, Volume 3, 55–63 (2021)
Time-dependent CME

\[ \vec{J}(\omega) = \sigma(\omega) \vec{B}(\omega), \quad \sigma_0 = \frac{e^2}{2\pi^2 \mu_A} \]

\[ \sigma(\omega) \sim \sigma_0 - i\xi_5 \omega \]

\[ \xi_5 = -\frac{0.5}{\alpha_s^2 \log(1/\alpha_s)} \frac{\sigma_0}{T} \]

in pQCD (Jimenez Alba-Yee)

\[ \sigma(0) = \sigma_0 \] is fixed by chiral anomaly

But, \( \sigma(\omega > 0) \) is from more general effects from P-odd helicity

From PRD 95, 051901 (2017) by Kharzeev-Stephanov-Yee
Anatomy of Chiral Magnetic Effect

<table>
<thead>
<tr>
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<th>$\omega \ll \tau_R^{-1}$</th>
<th>$\omega \gg \tau_R^{-1}$</th>
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<tr>
<td>$J^{\text{EQ}}$</td>
<td>$\frac{g}{6}$</td>
<td>0</td>
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<tr>
<td>$J^{\text{KM}}$</td>
<td>$1 - \frac{g}{6}$</td>
<td>$1 - \frac{g}{6}$</td>
</tr>
<tr>
<td>$J^{\text{M}}$</td>
<td>0</td>
<td>$-\frac{g}{6}$</td>
</tr>
<tr>
<td>$J^{\text{total}}$</td>
<td>1</td>
<td>$1 - \frac{g}{3}$</td>
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</tbody>
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$\frac{d\vec{B}(t)}{dt}$

$\vec{J}_M = \nabla \times \vec{M}$

Magnetization Current

Faraday's Effect

$\vec{E}(t) \sim \frac{d\vec{p}}{dt}$

$\sim \frac{d\vec{S}}{dt}$

$\sim \frac{d\vec{M}}{dt}$

Helicity $g$-factor

(Kharzeev-Stephanov-Yee)
Chiral Magnetic Wave
(Kharzeev-Yee, Burnier-Kharzeev-Liao-Yee)

\[ \partial_t n + \nabla \cdot \vec{J} = \partial_t n \pm \frac{1}{4\pi^2\chi} \vec{B} \cdot \nabla n = (\partial_t + \nabla \cdot \vec{v}_\chi) n = 0 \]

Hydrodynamic wave of chiral charges with velocity \( \vec{v}_\chi = \pm \frac{1}{4\pi^2\chi} \vec{B} \)

Quadrupole of charges
(Gorbar-Miransky-Shovkovy, Burnier-Kharzeev-Liao-Yee)
Anomalous Transport

(Son-Surowka, Landsteiner, ...)

\[ \vec{J} = \sigma_B \vec{B} + \sigma_V \vec{\omega} \]

\[ \sigma_B = \frac{\mu}{4\pi^2}, \quad \sigma_V = \frac{\mu^2}{8\pi^2} \]

Chiral Vortical Effect

\[ \vec{P} = \sigma_B^e \vec{B} + \sigma_V^e \vec{\omega} \]

\[ \sigma_B^e = \frac{\mu^2}{8\pi^2}, \quad \sigma_V^e = \frac{\mu^3}{6\pi^2} \]

Time Reversal (T) relates \( \sigma_V(k, \omega) = \sigma_B^e(k, \omega) \)

(Shiyong Li-Yee)
Gribov's Picture of Anomaly

\[ p_z H = \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \]

Band-Crossing Chiral Zero Mode

Landau Levels with 2D density of states \( \frac{eB}{2\pi} \)

One Weyl Fermion \( \Psi_R \)
Spectral Flow for Anomaly

\[ E(p_z) \]

\[ \frac{dp_z}{dt} = eE \]

\[ \frac{dn_A}{dt} = \left( \frac{eB}{2\pi} \right) \frac{1}{2\pi} \frac{dp_z}{dt} = \frac{e^2}{4\pi^2} \vec{E} \cdot \vec{B} \]

2D DoS  1D DoS
Dirac/Weyl Semi-Metals

\[ H = \sigma^+(D_x - iD_y)^N + \sigma^-(D_x + iD_y)^N + \sigma_z D_z, \quad \vec{D} = \vec{p} - e\vec{A} \]

\[ (p_x + ip_y) \rightarrow (p_x + ip_y)^N \]

(N-Winding Map)

Chiral Anomaly:
\[ \frac{dn}{dt} = N \frac{e^2}{4\pi^2} \vec{E} \cdot \vec{B} \]

N = 4 Chiral Zero Modes
Toplogy of Generalized Spinors

A generalized spinor $H = P_1(\vec{k})\sigma_x + P_2(\vec{k})\sigma_y + P_3(\vec{k})\sigma_z$

with arbitrary functions $P_1(\vec{k}), P_2(\vec{k}), P_3(\vec{k})$ in momentum $\vec{k}$

In both Heat-Kernel method (Fujikawa's method) and Diagrammatic computation, it was proven that

$$Winding\ Number\ of\ the\ Map\ N_p = \frac{1}{4\pi^2} \int d^3\vec{k} \ det \left( \frac{\partial P(k)}{\partial k} \right) e^{-P^2(\vec{k})}$$

Chiral Anomaly: $\partial_\mu j^\mu = N_p \frac{1}{4\pi^2} \vec{E} \cdot \vec{B}$

where $N_p = \frac{1}{\pi^2} \int d^3\vec{k} \ det \left( \frac{\partial P(k)}{\partial k} \right) e^{-P^2(\vec{k})}$

: Winding Number of the Map $\vec{k} \rightarrow \vec{P}(\vec{k})$ (Piljin Yi-Yee)
Topology of Generalized Spinors

Winding Number $N_p = \text{Berry's Monopole at } |\mathbf{k}| \to \infty$

The spinor at $\mathbf{k}$ is $|\psi(\mathbf{P}(\mathbf{k}))\rangle$ where

$$(\mathbf{P} \cdot \sigma) |\psi(\mathbf{P})\rangle = |\mathbf{P}| |\psi(\mathbf{P})\rangle$$

The Berry's monopole of $|\psi(\mathbf{P}(\mathbf{k}))\rangle$ in $\mathbf{k}$-space is $N_p$ times of unit monopole in $\mathbf{P}$-space

$$A_{ki} = \left( \frac{\partial P_j}{\partial k^i} \right) A_{pj}$$

$k = 0$ where Anomaly happens

Large $\mathbf{k}$ where Chiral Kinetic Theory is valid

(Son-Yamamoto, Stephanov-Yin, Chen-Pu-Wang-Wang)
UV-IR Connection

Index Theorem: \[ \text{Index} (D \cdot \sigma) = \frac{N_P}{8\pi^2} \int d^4x \ F \wedge F \]

Anomaly: Defined in Infrared (IR)

Chiral Anomaly: \[ \partial_\mu j^\mu = N_p \frac{1}{4\pi^2} \vec{E} \cdot \vec{B} \]

Anomaly: Defined in Infrared \( \vec{k} = 0 \) (IR)

Local Topological Density: Defined Locally in Space (UV)

Berry's Monopole defined in \( \vec{k} = \infty \) (UV)

Answers the question by Fujikawa and Mueller-Venugopalan
Spin of Magnetic Vortices

Superfluid Vortices

\[ L_z = \hbar N \quad , \quad N = \text{Particle Number} \]

Magnetic Vortices

\[ \Pi_\varphi = p_\varphi - qA_\varphi : \text{Gauge Invariant} \]

\[ L_z = r \times \Pi_\varphi : \text{Not a multiple of } \hbar \]

Special feature of

Representation of 2D Rotation Group \( \text{SO}(2) = \text{U}(1) \)
Feynman's Angular Momentum Paradox

Initial angular momentum $L_{z}^{\text{total}} = 0$

\[
\frac{d\Phi_B}{dt} = \vec{\nabla} \times \vec{E}
\]

\[
\vec{E} = \frac{dA}{dt}
\]

Faraday

\[
L_{z}^{\text{matter}} + L_{z}^{EM} = 0
\]

\[
L_{z}^{\text{matter}} = r \times \Pi_\phi = r \times (P_\phi - qA_\phi) = -q(r \times A_\phi) : \text{Not a multiple of } \hbar
\]

\[
L_{z}^{EM} = r \times (\vec{E} \times \vec{B}) \neq 0 : \text{Angular momentum of EM fields}
\]

Gauss' Law

$\vec{E}_{\text{radial}} \neq 0$

Poynting

$\vec{E} \times \vec{B} \neq 0$
Relativistic Magnetic Vortices

Neutral Nielsen-Olesen Vortex

Angular momentum from particle vortex and anti-particle anti-vortex cancel

Angular momentum is zero for a neutral vortex, where \( D_0 \Phi = 0 \),
\[
\vec{P} \propto i((D_0 \Phi)^* (\vec{D} \Phi) - \text{h.c.}) = 0
\]

Neutral vortex is a composite of particle vortex and anti-particle anti-vortex
\( \Phi \sim e^{i\phi} \sim a + b^\dagger \)

Particle-Vortex Duality in (2+1)D

\[
|D\Phi|^2 + V(|\Phi|^2) + \frac{1}{4}F_{\mu\nu}^2 \quad \leftrightarrow \quad (\partial\phi)^2 + W(|\phi|^2)
\]

Magnetic Vortex

Scalar Particle
For a charged vortex, there is a non-trivial cancellation $L_{z,\text{matter}} + L_{z,\text{EM}} = 0$

Non-Abelian Vortex in Color-Flavor-Locking (CFL) phase of dense 3-flavor quark matter $\langle q_\alpha q_\beta \rangle_{L, R} = \epsilon_{\alpha\beta\gamma} \epsilon^{ij\gamma} \Delta_{L, R}$

$U(1)_B \times SU(3)_L \times SU(3)_R \times SU(3)_c \rightarrow SU(3)_V$ : CFL is similar to Hidden Local Symmetry breaking for massive $\rho$-Mesons and massless Pions

A composite of Baryon Superfluid Vortex + Color Magnetic Vortex

It matches to Hadronc Phase
Vortices in Topological Insulator Surface

(Nogueira-Nussinov-Brink)

QHE of level \( \frac{1}{2} \):

\[ j^\mu = -\frac{e^2}{8\pi\hbar} \epsilon^{\mu\nu\alpha} F_{\nu\alpha} \]

In components:

\[ Q = \frac{e^2}{4\pi\hbar} B_z, \quad j_x = \frac{e^2}{4\pi\hbar} E_y, \quad j_y = -\frac{e^2}{4\pi\hbar} E_x \]

Lorentz force is zero: Zero Torque on TI Matter

\[ F_\phi = (Q \vec{E} + \vec{j} \times \vec{B})_\phi = QE_\phi - j_r B_z = 0 \]

The EM part \( L_{z}^{\text{EM}} \)

\[ L_{z}^{\text{EM}} = \int d^3r \ r \times (\vec{E} \times \vec{B}) = -\frac{e^2}{16\pi^2\hbar} \Phi_B^2 = -\frac{\nu^2}{16\hbar} \]

No TI matter part \( L_{z}^{\text{matter}} = 0 \)

Fractional Angular Momentum

\[ \frac{2e}{\hbar} \Phi_0 = 2\pi\nu \] for Superconductor Vortex
Spin Hydrodynamics and Pseudo-Gauge Transformations

(Shiyong Li-Stephanov-Yee)

Global Equilibrium with Angular Momentum Conservation

\[ \Delta E_1 = - \Delta E_2 \]
\[ \Delta N_1 = - \Delta N_2 \]
\[ \Delta J_1 = - \Delta J_2 \]

\[ dS = \beta dE - \alpha dN - \beta \vec{\Omega} \cdot d\vec{J} \]

Spin Potential

\[ \vec{J} = \vec{x} \times \vec{p} + \vec{S} : \text{Angular Momentum} \]

\[ dS = \beta (dE - \vec{v} \cdot d\vec{p}) - \alpha dN - \beta \vec{\Omega} \cdot d\vec{S} \]

\[ \vec{v} = \vec{\Omega} \times \vec{x} \quad , \quad \vec{\Omega} = \text{Fluid Vorticity} \]

Spin correction to 1st Law of Thermodynamics

(Hattori-Hongo-Huang-Matsuo-Taya)
Canonical EM Tensor and Spin

**Total Angular Momentum Tensor**

\[ J_{\mu\nu\alpha} = x^\nu T_{\mu\alpha} - x^\alpha T_{\mu\nu} + S_{\mu\nu\alpha} \]

Conservation of Angular Momentum \( \partial_\mu J_{\mu\nu\alpha} = 0 \) gives

\[ T_{\mu\nu} - T_{\nu\mu} = -\partial_\alpha S_{\alpha\mu\nu} \]

Canonical Energy-Momentum Tensor

It describes angular momentum exchange between spin and orbital angular momenta.
Spin Hydrodynamics

Canonical EM Tensor

\[ T_{C}^{\mu \nu} = (\varepsilon + p)u^{\mu}u^{\nu} + pg^{\mu \nu} + u^{\mu}q^{\nu} + u^{\nu}q^{\mu} - \frac{1}{2} \partial_{\alpha} S_{\alpha \mu \nu} \]

Anti-Symmetric Part

\[ j^{\mu} = nu^{\mu}, \quad s^{\mu} = su^{\mu} + \Delta s^{\mu} \]

The 2nd Law of Thermodynamics

\[ \partial_{\mu} S_{\mu} = \partial_{\mu} (\Delta s_{\mu} - \frac{1}{2} \beta_{\nu} \partial_{\alpha} S_{\alpha \mu \nu} - \beta q^{\mu}) + (-\beta a_{\mu} + \partial_{\mu} \beta)(q^{\mu} - \frac{T}{2} S_{\mu \nu} \partial_{\nu} \beta) = 0 \]

Thermal Hall Effect: \[ q^{\mu} = \frac{T}{2} S_{\mu \nu} \partial_{\nu} \beta \]

Ideal Limit of Spin Hydrodynamics

N.B. A more complete list of terms can be found in Gallegos-Gursoy-Yarom '21
Pseudo-Gauge Transformations
(Becattini-Tinti, Florkowski-Kumar-Ryblewski, Speranza-Weickgenannt)

Conservation Laws are not modified by

\[ \tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\alpha (\Sigma^{\alpha\mu\nu} - \Sigma^{\mu\alpha\nu} - \Sigma^{\nu\alpha\mu}), \quad \tilde{S}^{\mu\nu\alpha} = S^{\mu\nu\alpha} - \Sigma^{\mu\nu\alpha} \]

Magnetization current
\[ \vec{j} = \nabla \times \vec{M}, \quad \vec{M} = \frac{a}{2} \vec{\omega} \]

A special choice \( \Sigma^{\mu\nu\alpha} = S^{\mu\nu\alpha} \) makes \( \tilde{T}^{\mu\nu} \) symmetric Belinfante form and \( \tilde{S}^{\mu\nu\alpha} = 0 \), i.e. no spin tensor in theory

We get to the conventional relativistic hydrodynamics with symmetric EM tensor with no spin degrees of freedom

Barnett's Effect
\[ \vec{j} = \nabla \times \vec{M}, \quad \vec{M} = \frac{a}{2} \vec{\omega} \]
Physics of Pseudo-Gauge Transformations

\[ A_\mu \rightarrow \vec{M} \]

\[ \overline{J}_m = \nabla \times \vec{M} \]

\[ \overline{g}_{\mu\nu} \]

\[ \overline{P}_m = \frac{1}{2} \nabla \times \vec{S} \]

Magnetization Current

``Spinetization`` Momentum
Non-Dissipative Second Order Transports

As a result of pseudo-gauge transformation to get to symmetric Belinfante EM tensor, we obtain several Non-Dissipative Second Order Transport coefficients related to Fluid Vorticity $: \partial_\mu \tilde{s}^\mu = 0$:

$$\tau^{\mu\nu} = \frac{\chi}{2} (\sigma_\alpha^{\mu} + \omega_\alpha^{\mu}) \omega^{\alpha\nu} + (\mu \leftrightarrow \nu) + 2a_0 \Delta^{\mu\nu} \omega_\nu \omega^\nu$$

$$\tau^\mu = -\frac{T_n}{2w} \Delta_\lambda^\mu \partial_\nu (\beta \chi \omega^\lambda^\nu) - \frac{1}{2} \Delta_\lambda^\mu \partial_\nu (a \omega^\lambda^\nu)$$

$$\Delta s^\mu = -\frac{TS}{2w} \Delta_\lambda^\mu \partial_\nu (\beta \chi \omega^\lambda^\nu) - \frac{1}{2} \Delta_\lambda^\mu \partial_\nu (b \omega^\lambda^\nu) + \frac{n\chi}{2w} \omega^{\mu\nu} \partial_\nu \alpha$$

It is shown that the thermodynamics can also be made conventional

$$\tilde{S}(\tilde{\varepsilon}, \tilde{n}, \omega^\mu) = s(\tilde{\varepsilon} + \Delta \varepsilon, \tilde{n} + \Delta n, \omega^\mu) - \Delta s$$

$$d\tilde{s} = \tilde{\beta} d\tilde{\varepsilon} - \tilde{\alpha} d\tilde{n} , \quad \tilde{s} = \tilde{\beta} (\tilde{\varepsilon} + \tilde{p}) - \tilde{\alpha} \tilde{n}$$
Why do we have these equivalent descriptions?

In fact, there are infinitely many equivalent descriptions by performing partial pseudo gauge transformations with

\[ \Sigma^\mu_{\nu\alpha} = tS^\mu_{\nu\alpha}, \quad 0 < t < 1. \] The \( t = 1 \) is the special point where the energy-momentum tensor becomes the symmetric Belinfante form

\[ \partial_\mu F^\mu_{\nu} = J^\nu_{\text{total}} = J^\nu_{\text{hydro}} - \partial_\alpha \left( \frac{a}{2} \omega^\nu_{\alpha} \right) \]

Hydrodynamics, a priori, does not distinguish between \( J^\mu_{\text{total}} \) and \( j^\mu_{\text{hydro}} \)

\[ G^\mu_{\nu} = 16\pi G_N T^\mu_{\nu B} = 16\pi G_N (T^\mu_{\nu \text{hydro}} + \frac{1}{2} \partial_\alpha (S^{\alpha\mu\nu} - S^{\mu\alpha\nu} - S^{\nu\alpha\mu})) \]
Equivalent Hydrodynamics

Given a microscopic theory with several constituents carrying spins, the macroscopic hydrodynamics may choose the canonical EM tensor for some degrees of freedom and the Belinfante EM tensor for other degrees of freedom.

All these discrete choices are equivalent hydrodynamic descriptions of the same microscopic theory.
Universality of Hydrodynamics comes with Generosity

The origin of pseudo gauge transformation is the microscopic equivalence between spin and "spinetization" momentum, that hydrodynamics can not resolve macroscopically. To accommodate such microscopic equivalences for any system, that hydrodynamics can not probe in macroscopic scales, what hydrodynamics can do is to allow much more generous equivalences with continuous parameters of pseudo gauge transformation, to be able to accommodate any system.
Thank you!