Semi-abelian gauge theories, non-invertible symmetry & string tension beyond N-ality.

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MOTIVATION

- There are a handful of QFTs in 3d $\mathbb{R}^4$ where confinement and mass gap generation can be understood analytically.

  Polyakov model on $\mathbb{R}^3$ (75)
  Seiberg–Witten on $\mathbb{R}^4$ (94)
  QCD(adj) on $\mathbb{R}^3 \times S^1$
  deformed Yang–Mills on $\mathbb{R}^3 \times S^1$ (08, w/ Yaffe, & w/ Shifman)

- In all of the above, $\text{SU}(N) \rightarrow \text{U}(1)^{N-1}$ due to either an adjoint Higgs vev (algebra valued) or gauge holonomy (group valued)

  "Dynamical abelianization"

- In all, $S_N = \text{Weyl group of } \text{SU}(N)$ is Higgsed.

  $\xrightarrow{\text{permutation group}}$

- Higgsing of $S_N$ pervades the physics of these systems & renders them quite distinct from pure YM.
Spectrum of fundamental string tensions. \( N-1 \) types in SW & Polyakov. (see Douglas, Shenker 95) instead of one!

Spectrum of "glueball" masses. \( N-1 \) distinct dual photon masses in all abelianizing theories.

\[
\begin{array}{c}
\text{mass} \\
m_{N-1} \\
\vdots \\
m_1 \\
\end{array}
\]

while in YM.

\[
\begin{array}{c}
\text{mass} \\
glueball \\
\text{gap} \\
\end{array}
\]

Splitting to \( N-1 \) \( \implies \) Imprint of broken \( S_N \).

Tensions dictated by charges \( w \in \mathbb{Z} \) \& \text{not} \( G \) (gave it lattice) (sometimes presented as deficiency. Not so, it is a feature).
**Curious Fact about String Tension in YM.**

- In YM or QCD with adjoint matter, we are usually told that tensions are dictated by $N$-ality. This is true.

- However, the real story is actually more interesting.

\[
\Psi_R \quad \leftrightarrow \quad \overline{\Psi}_R \quad \rightarrow \text{representation}
\]

- Based on lattice: Bali (93), de Forcrand, Philipsen (99), Wipf et al. (10), Greensite (83), ...

  *(not widely appreciated)*
\[
R = F_j F \otimes F, \ldots, F \otimes \ldots \otimes F
\]
\[n \leq \lfloor \frac{n}{2} \rfloor\]

\[
R = \text{adj, adj} \otimes \text{adj}, \ldots
\]

\[
R = F \otimes \ldots F \otimes \ldots F
\]

WHY?
DEFINITION OF SEMI-ABELIAN THEORY.

- We construct two interesting gauge theories.
  - $U(1)^{N-1}$ theory with global non-abelian discrete sym. $S_N$
  - $U(1)^{N-1} \times S_N$ semi-abelian g.t. obtained by gauging $S_N$. (coupling to $S_N$ TQFT)

Semi-direct product.

Abelianizing theories $SU(N) \rightarrow U(1)^{N-1}$

- $S_N$: Higgsed

Semi-abelian

- $U(1)^{N-1} \times S_N$
- $S_N$: not Higgsed

Non-abelian $SU(N)$ theories.

- $S_N$: not Higgsed

- Many properties of semi-abelian theory are much closer to pure YM than abelianizing theories.
3d LGT

- Wilson formulation \{ of 3d U(1)^{N-1} Lattice g.t. (LGT) \\
- Villain formulation \}

- Equivalent at weak coupling.
  - Villain more convenient for exact dualities.
  - Wilson better for gauging $S^N$.

We work with 3d LGT; very concrete and simple illustration of some general ideas.
\( \Lambda_3 \): Original lattice

\( C(r) \): \( r \)-cells, \( r = 0, 1, 2, 3 \)
- site, link, plaquette, cube.
\[ \{ s, l, p, c \} \]

\( \tilde{\Lambda}_3 \): Dual lattice.

\[ *C(r) = \tilde{C}(d-r) \]

\( *s = \tilde{c} \)
\( *l = \tilde{p} \)
\( *p = \tilde{l} \)
\( *c = \tilde{s} = x = x + \frac{1}{2}(1+2+3) \)

\( \tilde{s} = \tilde{x} \)
\( \tilde{l} \)
\( \tilde{p} \)

shifted by half.
Wilson formulation

\[ S_W = \beta \sum_{p} \sum_{i=1}^{N} (1 - \cos f_p^i) - i \sum_{e} \sum_{i=1}^{N} a_e^i \]

\[ f_p^i = (d a^i)_p \quad ; \quad a_e^i \in [0, 2\pi) \]

Lagrange multiplier. Produce constraint \( \sum_{i=1}^{N} a_e^i = 0 \).

- Constraint tells us that only \( N-1 \) photons are physical.

- Global \( S_N \) permutation symmetry manifest.

\[ (a_e^1, \ldots, a_e^N) \stackrel{P \in S_N}{\rightarrow} (a_e^{P(1)}, \ldots, a_e^{P(N)}) \]

- This is the first example of a pure gauge theory that is equipped with non-abelian global symmetry. (up to my knowledge)
Villain formulation.

\[ S = \frac{1}{4\pi^2} \sum_p \left( F_p + 2\pi n_p \right)^2, \quad F_p = (dA)_p \]

- For a valued in \( \mathbb{R}^{N-1} \)
- \( n_p \) valued in \( \Gamma \subset \mathbb{R}^{N-1} \)

\[ \Gamma_r : \text{Root lattice of } SU(N). \]

0-form gauge invariance: \( A_e \rightarrow A_e + (d\lambda)_e \)

1-form gauge inv. \( A_e \rightarrow A_e + 2\pi \beta_e \)

\( n_p \rightarrow n_p - (d\beta)_p, \beta_e \in \Gamma \).

Since

\[ \mathbb{R}^{N-1} / 2\pi \Gamma \cong U(1)^{N-1} \]

this also defines a \( U(1)^{N-1} \) LGT.

\[ Z = \sum \int [dA_e] e^{-S} \]

\( \{ n_p \in \Gamma \} \subset \mathbb{R}^{N-1} \)

- \( \text{Action in Villain form; can be exactly dualized.} \)
Global symmetries (formal)

- Improved recent understanding of p-form symmetries.
  
  Symmetry: \( \text{def} \) existence of topological generators \( U(M^{d-1}) \)
  
  (Gaiotto, Kapustin, Seiberg, Wilket '04)

- Textbook case (\( p=0 \))
  
  \( Q = \int_{\text{space}} J_0 \rightarrow \text{consider} \ Q = \int_{M^{d-1}} \ast J = Q(M^{d-1}) \)
  
  change on any \((d-1)\) manifold \(M^{d-1}\).

\( U_\alpha(M^{d-1}) = e^{idQ(M^{d-1})} \)

\( \alpha \): Group label.

- \( U_\alpha(M^{d-1}) \) implements symmetry action on charged operators.

\( p=1 \) 1-form center sym.

- Center sym. is a 1-form symmetry (acting on lines) generated by co-dim 2 defect.

\[ W_R(C) \rightarrow e^{\frac{2\pi i}{N}|R|} W_R(C) \]

- \( |R| \) = N-ality.
  
  Nice explanation of N-ality rule.
Global symmetries

- 1-form symmetry: \( A_e \rightarrow A_e + \theta e_j \quad \theta \in \mathbb{R}^{N-1} \quad (d\theta)_j = 0 \)

- Discrete non-abelian 0-form symmetry:

  \( A_e \rightarrow \pi A_e \quad \eta_p \rightarrow \pi \eta_p \)

\( \Pi: O(N-1) \) transformations that preserve root lattice \( \Gamma_r \).

Weyl group of \( SU(N) \), \( S_N \) & \( \mathbb{Z}_2 \) charge cons.

\( A_e \rightarrow -A_e \)

\( \eta_p \rightarrow -\eta_p \)

\( G^{[0]} = \begin{cases} S_N \times \mathbb{Z}_2 & N \geq 3 \\ \mathbb{Z}_2 & N = 2 \end{cases} \quad G^{[1]} = \{ U(1)^{N-1} \} \)
Weyl reflections: $S_2(\mathbf{v}) = \mathbf{v} - \alpha (\alpha \cdot \mathbf{v})$

$S_3$

permutation group, ($S_N$ sym of $N-1$ simplex)

$\mathbb{Z}_2$ charge conjugation:

$G^{[0]} = S_3 \times \mathbb{Z}_2$
Observables.

- Wilson lines: \( W_\omega (c) = e^{i \int c \cdot A} \)

Invariance under 1-form gauge transformation \( \Rightarrow \omega \in \Gamma_w \)

- Transformation under global symmetries:
  
  \[
  G^{[0]} : W_\omega (c) \rightarrow W_{\pi \cdot \omega} (c).
  \]
  
  \[
  G^{[1]} : W_\omega (c) \rightarrow W_\omega (c) e^{i \int c \cdot \theta}.
  \]

- Contractible loops are invariant under \( G^{[0]} \).
- Non-contractible loops transform.
Villain $\Rightarrow \Gamma_w$-ferromagnet $\Rightarrow$ Magnetic Coulomb gas

- Poisson resummation,
  $\sum_{\mathbf{q} \in \Gamma_F} e^{-\frac{1}{4\pi e^2} (F_\mathbf{q} + 2\pi m_{\mathbf{q}})^2} = \sum_{\mathbf{k} \in \Gamma_w} e^{-\pi e^2 k_\mathbf{k}^2 + i \mathbf{k} \cdot F_\mathbf{q}}$

- Integrate out $A_\mathbf{q}$ exactly, $\Rightarrow (d^k k)_e = 0$. $\Rightarrow$
  $(x_\mathbf{k})_e = (d^m m)_e$ where $m_{\widetilde{x}}$ is $\Gamma_w$-valued scalar on the dual lattice $\widetilde{\Lambda}_3$.

$Z = \sum_{\{m_{\widetilde{x}} \in \Gamma_w\}} e^{-\pi e^2 \sum_{\widetilde{e}} (d^m m)_\widetilde{e}^2}$

$\Gamma_w$-ferro.
Exact dual of Villain.
Let us replace \( m(\vec{x}) \) with a continuous field. Using another Poisson resummation:

\[
\sum_{m(\vec{x}) \in \Gamma_m} \delta(\sigma(\vec{x}) - 2\pi m(\vec{x})) = \sum_{q(\vec{x}) \in \Gamma_r} e^{i q(\vec{x}) \cdot \vec{\sigma}(\vec{x})}
\]

\[
Z = \int [d\sigma(\vec{x})] \sum_{\{q(\vec{x}) \in \Gamma_r\}} e^{-\frac{e^2}{4\pi} \sum_{\vec{x}} \left( \partial_{\mu} \sigma(\vec{x}) \right)^2 + i \sum q(\vec{x}) \cdot \vec{\sigma}(\vec{x})}
\]

\[
q(\vec{x}) = \oint_{\text{faces, center } \vec{x}} n_\mathbf{p} \quad \in \Gamma_r
\]

\( \rightarrow \) Magnetic charge.

\( \rightarrow \) Gauss surface.
Coulomb gas representation.

- Integrating out $\Gamma(\tilde{x})$ exactly, we reach to Coulomb gas representation.

$$Z = \sum_{\{\tilde{q}(\tilde{x}) \in \Gamma_f\}} e^{-\frac{\pi}{e^2} \sum_{\tilde{x}, \tilde{x}'} \nu(\tilde{x} - \tilde{x}') \tilde{q}(\tilde{x}) \cdot \tilde{q}(\tilde{x}')$$

- Euclidean Vacuum: Proliferation of monopoles, but very different from Polyakov model & deformed YM due to unbroken $Sp$. 

Effective Field Theory.

- \( L = -\frac{e^2}{4\pi} \sum_{\tilde{x}} \sigma(\tilde{x}) \Delta \sigma(\tilde{x}) + 2 e^{-\frac{1}{2} T} \sum_{\tilde{x}} \sum_{\alpha \in \Phi} \cos (\alpha \cdot \sqrt{s} \tilde{x}) \)

\( \Phi \): all \( N^2 - N \) roots of \( SU(N) \) algebra. \( \alpha \in \text{Adj} (SU(N)) \)

All on the same footing due to unbroken \( SU(N) \).

In Polyakov model where \( SU(N) \to U(1)^{N-1} \) actions are hierarchical.

Hierarchical (Polyakov)

Egalitarian (Semi-abelian)

\( x_1, x_2, \ldots, x_{N-1} \): \( e^{-\frac{1}{2} s} \)

\( x_1 + x_2 + x_3, \ldots, x_{N-1} + x_N \): \( e^{-\frac{3}{2} s} \)

\( x_1 + x_2 + x_3 \): \( e^{-\frac{5}{2} s} \)

\( x_1 + \cdots + x_{N-1} \): \( e^{-(N-1)s} \)

All \( N^2 - N \) roots \( \alpha \in \Phi \) have the same action. \( \text{Adj} (SU(N)) \).
Dramatic differences

- In Polyakov: there are \( N-1 \) dual photon mass \& \( N-1 \) fundamental string tension. (same in SW \( N=2 \))

- In \( U(1)^{N-1} \) LGT with \( S_N \):
  - \( m \) \( \Rightarrow T_{m_1} = T_{m_1-d_1} = \cdots = T_{m_1-d_1-\cdots-d_{N-1}} \)
  - \( N-1 \) fold deg. \& Imprints of unbroken \( S_N \).

- Imprints of broken(Higgsed) \( S_N \).
Wilson loops & string tensions.

- Let $\omega \in \Gamma_w$ be electric charge & $C = \partial D$ be a contractible loop.

\[
W_\omega(C) = e^{i\oint_C \omega \cdot A} \to \text{Line integral.}
\]

\[
= e^{i\oint_S \omega \cdot F} = e^{i\sum_{\text{Poincaré dual}} \langle \omega, F_p \rangle}
\]

- $W_\omega(C)$: Defect operator in dual formulation. Delete $C$ & restrict path integral to config. $\oint_{S'} d\sigma = 2\pi \omega \in 2\pi \Gamma_w$.

$\text{link}(S', C) = 1$. 

$[\text{D}]$: Poincaré dual, bump one form function equal to 1 on $\text{supp}$, 0 otherwise.
\[ \langle W_\omega (C) \rangle = \int D\sigma \ e^{-\frac{e^2}{2\pi^2} \int d^3x \left( \frac{1}{2} |d\sigma - 2\pi \omega [D]|^2 + M^2 \sum_{\alpha \in \Phi^1} (1 - \cos \alpha \cdot \sigma) \right)} \]

\[ = e^{-T_\omega \ Area(D)} \]

- Area law of confinement.
- \( T_\omega \neq 0 \) string tension

- Action of \( S(z) = S = Area[D] \times T_\omega \)
  where \( T_\omega \) is instanton action in reduced QM system.
  - \( A[D] \) is area.

- Equivalent to instanton calculation in QM

\[ T_\omega = \min_{\sigma(z)} \frac{e^2}{2\pi} \int_0^\infty dz \left( \frac{1}{2} \left( \frac{d\sigma}{dz} \right)^2 + M^2 \sum_{\alpha \in \Phi^1} \right) (1 - \cos \alpha \cdot \sigma) \]
\[ \{ \mu_1, 2\mu_1, \mu_2, x \} \in \Gamma_w \text{ highest weights of } \{ F, S, \text{adj} \} \text{ reps of } SU(N). \]

\[ T_{\mu_1}, T_{2\mu_1} = 2 T_{\mu_1}, \quad T_{\mu_2} = \frac{2(N-2)}{N-1} T_{\mu_1}, \quad T_{\mu} = 2 T_{\mu_1} \]

- If \( \omega_1 \neq \Omega \omega_2 \) generically, \( T_{\omega_1} \neq T_{\omega_2} \).

- There is no \( \omega \in \Gamma_w \) for which \( T_{\omega} \) vanishes. This simple fact will be important later.
Gauging $S_n$ is good.

- Clearly, properties of our abelian theory with $S_n$ global symmetry resembles to YM more than dynamically abelianizing theories.
- But global symmetries are different from YM.

\[ U(1)^{n-1} \text{ with global } S_n \]
\[ G^{(0)} : S^n \times \mathbb{Z}_2 \]
\[ G^{(1)} : U(1)^{n-1} \]  
\[ \text{charge cong.} \]

\[ \frac{SU(n)}{\mathbb{Z}_2} \]
\[ G^{(0)} : \mathbb{Z}_2 \]
\[ G^{(1)} : \mathbb{Z}_N \]

Let us gauge $S_n$, and construct $U(1)^{n-1} \times S_n$ theory. Then, we will show that

\[ \frac{U(1)^{n-1} \times S_n}{\mathbb{Z}_2} \]
\[ G^{(0)} : \mathbb{Z}_2 \]
\[ G^{(1)} : \mathbb{Z}_N \]
Semi-abelian gauge theory.

- Start with Wilson formulation and gauge $S_N$.
- Embed elements of $U(1)^{N-1} \times S_N$ inside $SU(N)$ as
  $$P \cdot C \in SU(N) \quad \Rightarrow \quad C = \begin{pmatrix} e^{i\alpha_p} & \ast \\ \ast & e^{i\alpha_N} \end{pmatrix}$$

  $P \in S_N$: $N \times N$ matrix rep. of Weyl reflection.

- $(P_1 \cdot C_1)(P_2 \cdot C_2) = P_1 P_2 \cdot (P_2^{-1} C_1 P_2 C_2)$

- Let $(P_e \cdot C_e) \in SU(N)$ denote link variable.

- $S = \sum_P \beta_1 \left( \text{tr}(1 - \prod_{\text{loop}} (P_e \cdot C_e)) + \sum_{P} \beta_2 \text{tr}(1 - \prod_{\text{loop}} P_e) \right)$
**Coupling to Sn TQFT.**

- $\beta_2 \to \infty \Rightarrow$ imposes flatness condition on $S_N$ gauge field.
- $L_{S_N}$ monopoles are energetically forbidden.
- $\prod n = 1 \Rightarrow S_N$ gauge fields become topological.

As long as correlators do not involve non-contractible cycles on $M_3$, local dynamics must be the same as abelian model.

- $\Phi$ can be fixed to 1 inside $B^3$.
- Local dynamics same as abelian model.

$$\langle \phi_1(x) W^{S_N} \phi_2(x) \rangle_{\partial_1 \times S_N}^{U(1) \times U(1)} = \langle \phi_1(x) \phi_2(x) \rangle_{\partial_1}^{n-1}$$

Globally different theories, e.g. correlators involving non-contractible cycles know about $S_N$ TQFT.
Wilson loops & symmetry generators in abelian $U(1)^{N-1}$ model.

Wilson loops: $W_k(C) = e^{i \left( \sum_{j=1}^{k-1} \alpha_j \right) \int_{C} A}$ \hspace{1cm} \text{for } k = 1, \ldots, N.

$W_2 \leftrightarrow m_1 - d_1$

$W_1 \leftrightarrow m_1$

$W_3 \leftrightarrow m_1 - d_1 - d_2$

$[U(1)]^{N-1}$ generators: $U_{\theta}^{(k)}(C) = e^{i \frac{\theta}{2\pi} \int_{C} A_k \cdot d\sigma}$ \hspace{1cm} \text{for } k = 1, \ldots, N-1.$

Eigenvalue eq.

$W_i(C_2)$: Eigen-operator.

$U_{\theta}(C_1)$: Topological line operator (generator).

$U_{\theta}(C_1)$ = $U^{(k)}(C_1)$ = $e^{i \theta \delta_{k1}}$
Gauging $S_n$

- After gauging $S_n$, generators & Wilson operators are no longer gauge invariant.
- We must symmetrize Wilson loops & topological operators to build gauge invariant objects.

Wilson loops in $U(1)^{N-1} \times S_n$ theory:

$$W_{fd}(c) = W_1(c) + W_2(c) + \cdots + W_N(c)$$

$\mathbb{Z}_N$ generators:

$$U_n(c) = \prod_{k=1}^{N-1} U^{(k)}_{\frac{2\pi}{N}kn}(c) = e^{i \frac{n}{N} \int_c (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{N-1}) \cdot dr}$$

$$U_n(c) U_m(c) = U_{n+m \mod N}(c)$$

Fusion obeys group multiplication laws.
\[ \mathbb{Z}_N^{[1]} \text{ center & failure of } N\text{-ality rule.} \]

\[ \mathbb{Z}_N^{[1]} : W_{fd} \rightarrow e^{2\pi i/N} W_{fd}, \quad \text{or} \]

\[ \langle U(C_1) W_{fd}(C_2) \rangle = e^{\frac{2\pi i}{N} \text{Link}(C_1, C_2)} \langle W_{fd}(C_2) \rangle \]

\[ \begin{array}{c|c|c}
W_{fd}(C_2) & \text{adj. } G & \text{adj. } G \\
\text{U}(C_1) & 1 & 1 \\
\end{array} \]

**Q:** Does 1-form \( \mathbb{Z}_N^{[1]} \) center sym. control string tensions?

No. \( T_{\mu} \neq 0 \), but \( T_\alpha \neq 0 \) as well (zero N-ality). \( T_{2\mu_1} \neq T_{2\mu_2} \) (both N-ality 2).

- Infinitely many string tensions, instead of just \( N \).

**Puzzle:** How is the presence of infinitely many string tensions compatible with finite center symmetry?
Non-invertible topological lines & symmetry.

- Need to explain the failure of the N-ity rule.
- Non-invertible top. symmetry. Generalization of 1-form $\mathbb{Z}_N$ center symmetry.
- Symmetrizing center generators $U^{(1)}_\theta(C)$ in abelian theory:

$$T_\theta(C) = \frac{1}{N!} \sum_{P \in S_N} P U^{(1)}_{\theta}(C) P^{-1}$$

$$= \frac{1}{N(N-1)} \sum_{\alpha \in \Phi} e^{i \frac{\theta}{2\pi} \int_C \alpha \cdot d\tau}$$

- Satisfies all features of 1-form symmetry except group multiplication law.

$$T_\theta(C) T_{\theta'}(C) \neq T_{\theta+\theta'}(C)$$

Symmetry, but not a group.
Consider Wilson loop that corresponds to \( \omega \in \Gamma_w \).

\[
W_R = \frac{1}{N!} \sum_{P \in \mathcal{P}} P W \omega P^{-1} \quad \text{etc.}
\]

\[
T_\theta (c) \ W_R (c') = \lambda_{\theta,R} \ W_R (c')
\]

\[
\lambda_{\theta,R} = \frac{1}{N(N-1)} \sum_{\alpha \in \Phi} e^{i \theta \alpha \cdot \omega}
\]

\[
T_\theta (c) \ W_{fd} (c') = \frac{1}{N} (N - 2 + 2 \cos \theta) \ W_{fd} (c')
\]

2d QFTs: cf Bhardwaj, Tachikawa (17), Buican, Gromov (17), Thorngren, Wang (19), ... Karmgrodski et al. (20).
Tension for $N$-ality zero rep.

- Here is the main point concerning non-invertible sym.
- Despite the fact that $W_{adj}$ is trivial under $Z_N^{(1,1)}$, \( (1)\ (C_1) \ W_{adj} (C_2) = 1 \ W_{adj} (C_2) \) it obeys

\[
T_\theta (C_1) \ W_{adj} (C_2) = \frac{(N-2)(N-3) + 4(N-2) \cos \theta + 2 \cos (2\theta)}{N(N-1)} \ W_{adj} (C_2)
\]

- String tension beyond $N$-ality is characterized by non-invertible topological line operators $T_\theta$.
- $T_\theta$ is a 1-form sym. But it does not have an inverse. $T_\theta \neq T_{\bar{\theta}}$. Fusion rule does not conform with group law.
- Fusion category???
Distinguishing two N-ality two reps.

- Consider two N-ality 2-reps, \( W_{\text{sym}} \neq W_{\text{asym}} \).
- Eigen-operators of non-invertible generators \( T_\theta (c_i) \) are
  \[ W_{\text{asym}} \neq (W_{\text{sym}} - W_{\text{asym}}) \.
  \]

  \[
  T_\theta (c_i) (W_{\text{sym}} - W_{\text{asym}})(c_2) = \frac{N - 2 + 2 \cos 2\theta}{N} \quad (W_{\text{sym}} - W_{\text{asym}})
  \]

  \[
  T_\theta (c_4) W_{\text{sym}}(c_2) = \frac{(N - 2)(N - 3) + 2 + 4(N - 2)\cos \theta}{N(N - 1)} \quad W_{\text{sym}}(c_2)
  \]

- Consistent with \( T_{\mu_2} \neq T_{\mu_1} \).
Add $W$-bosons. Since eigenvalue of $W_{adj}$ must be one, (perimeter law or adjoint string breaking);

$$T_{\Theta}(c_1) W_{adj}(c_2) = \lambda_{\Theta, adj} W_{adj}(c_2)$$

the only solution is for $\Theta=0$, and we loose non-invariance. (cf. Rudelius, Shao 20, discrete gauge theories.)

If $W$ is heavy, then:

- suspiciously curious, isn't it?
Conclusions - speculations

- Abelian and semi-abelian dynamics are quite different from other solvable, abelianizing theories: Polyakov, N=2 SW, QCD(adj), deformed YM. Provides new insights.

- Non-invertible sym. is very likely also present in pure YM as an approximate sym. We believe that it becomes exact at $N=\infty$ limit and string tensions are not controlled by $N$-ality.

- Non-inv. sym can very likely provide a meaning to confinement in theories such as QCD(F) and YM with $G_2$ group etc. It also gives a meaning to confinement of adjoint probes in SU(N) YM.