Electron hydrodynamics in solids

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Outline

1. Basics of hydrodynamics
2. Hydrodynamics in solids and experimental observations
3. Consistent hydrodynamics in Weyl semimetals and electron flows
Basics of hydrodynamics
Definition of hydrodynamics

- **Hydrodynamics** is the macroscopic theory that studies the motion of various fluids (including gases).

- Key variables:
  - Fluid velocity: \( u = u(t, r) \)
  - Particle density: \( n = n(t, r) \)
  - Energy density: \( \epsilon = \epsilon(t, r) \)

- Hydrodynamics is based on the conservation laws: momentum, mass, and energy.

[Images of Archimedes, Leonhard Euler, Daniel Bernoulli, Claude-Louis Navier, and George Stokes]
Key equations

- **Navier-Stocks equation**: 
  \[ n \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla P + \eta \Delta u + \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot u). \]

- Advection term
- Diffusion term
- Compressibility

\( \eta \) is the shear viscosity
\( \zeta \) is the bulk viscosity
Key equations

- **Heat transfer** equation:

\[ nT \left[ \partial_t s + (u \cdot \nabla)s \right] = \nabla_j \kappa \nabla_j T + \frac{\eta}{2} \left( \partial_j u_i + \partial_i u_j - \frac{2}{3} \delta_{ij} \partial_l u_l \right)^2 + \zeta (\nabla \cdot u)^2 \]

\( s \) is the entropy density \( \kappa \) is the thermoconductivity

- **Continuity** equation:

\[ \partial_t n + (\nabla \cdot J) = 0. \]

\( J = nu \) is the particle current density
Reynolds number and turbulence

- Reynolds number:
  \[ \text{Re} = \left| \frac{\eta \nabla_j \nabla_j u_i}{u_j \nabla_j u_i} \right| = \frac{n u L}{\eta} = \frac{u L}{\nu} \]

  \( \text{Re} \ll 1 \) **laminar** (layered) flow \quad \( \text{Re} \gg 1 \) **turbulent** (chaotic) flow

- Vortices and turbulence:

  von Kármán vortex street
Subfields of fluid dynamics

The number of subfields in fluid dynamics is numerous:
- Aerodynamics
- Magneto-hydrodynamics
- Geophysical fluid dynamics and meteorology
- Hemodynamics
- etc.

System scales range from fm to parsec.
Hydrodynamics in solids and experimental observations
Two main regimes


Nonhydrodynamic regimes: $l_{MR} \ll l_{MC}, L$, where $L$ is the sample size. [J. Zaanen, Science 351, 1058 (2016)]

Hydrodynamic regime: $l_{MC} \ll L \ll l_{MR}$. 

- Ballistic regime $l_{\text{eff}} \sim L$
- $R(T) \sim T^{-2}$ is affected by $e^-e^-$ collisions $l_{\text{eff}} \sim L^2 / l_{ee}$
- $R(T) \sim T^5$ stems from electron-phonon interactions
- $R(T) \sim \frac{1}{l_{\text{eff}}}$
Experimental observations


- Viscous contribution to the resistance of 2D metal PdCoO$_2$ (Poiseuille flow) [P.J.W. Moll et al., Science 351, 1061 (2016)]

  - Higher than ballistic transport in constrictions [H. Guo et al., PNAS 114, 3068 (2017); R. Krishna Kumar et al., Nat. Phys. 13, 1182 (2017)]
  - Visualization of the Poiseuille flow via the Hall field profile [J.A. Sulpizio et al., Nature 576, 75 (2019)]
Backflows in graphene and GaAs

Whirlpools

Negative potential regions


[D.A. Bandurin et al., Science 351, 1055 (2016)]

Electron flow through a constriction

**Theory**
[H. Guo et al., PNAS USA 114, 3068 (2017)]

**Experiment**
[R. Krishna Kumar et al., Nat. Phys. 13, 1182 (2017)]
Visualizing electron flow: Hall voltage

[J.A. Sulpizio, L. Ella, A. Rozen et al., Nature 576, 75 (2019)]

single electron transistor

$$E_y = \frac{B}{en} \left( j_x + \frac{t_{ee}^2}{2} \partial_y^2 j_x \right)$$

$$B = 12.5 \text{ mT}$$ \hspace{1cm} \text{ballistic} \hspace{1cm} n = -6 \times 10^{11} \text{ cm}^{-2} \hspace{1cm} \text{hydrodynamic}

Ballistic \hspace{1cm} T=7.5K

Hydrodynamic \hspace{1cm} T=75K
Visualizing electron flow: phase diagram

Momentum-relaxing mean-free path

$e^-e^-$ collision length

Curvature of the Hall voltage

Boltzmann theory
Dirac fluid and WF law violation

Wiedemann-Franz law:

\[ L \equiv \frac{\kappa_e}{\sigma T} = \frac{\pi^2}{3e^2} = L_0 \]

Electric transport is sensitive to the \( h^+ e^- \) collisions

Dirac fluid regime
Preturbulent regimes in graphene

\[ T \gg \mu \]
\[ \eta \approx 0.45 \frac{T^2}{4\hbar v_F^2 \alpha^2} \]
\[ u \approx 10^5 \text{ m/s} \]
\[ L \approx 5 \text{ \mu m} \]
\[ \nu_{\text{eff}} = \frac{v_F^2 \eta}{(Ts)} \]
\[ \approx 5 \times 10^{-3} \text{ m}^2/\text{s} \]
\[ \text{Re} = \frac{sT L u}{v_F^2 \eta} = \frac{uL}{\nu_{\text{eff}}} \]
\[ \omega_{\text{shedding}} \sim 0.1 \text{GHz} \]
Hydrodynamics in Weyl semimetals

- Weyl semimetals WP$_2$ [J. Gooth et al., Nat. Commun. 9, 4093 (2018)]

\[ \rho = \rho_0 + \rho_1 w^\beta \]

\[ L = \frac{\kappa \rho}{T}, \quad L_0 = \frac{\pi^2 k_B^2}{3 e^2} \]
Consistent hydrodynamic in Weyl semimetals

[E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, and P.O. Sukhachov, Phys. Rev. B 97, 121105(R) (2018); 97, 205119 (2018); 98, 035121 (2018)]
Low energy Weyl fermions

\[ H_0(k) = \begin{pmatrix} \nu_F \sigma \cdot (k - b) + b_0 & 0 \\ 0 & -\nu_F \sigma \cdot (k + b) - b_0 \end{pmatrix}. \]

Chiral shift parameter \(-\vec{b} \cdot \vec{\gamma} \gamma_5\)

[E.V. Gorbar, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. C 80, 032801(R) (2009)]
Berry curvature

- Consider the adiabatic evolution of a system [M.V. Berry, Proc. R. Soc. A 392, 45 (1984)]. At each time moment, the system is at its instantaneous eigenstate:
  \[ H = H(k), \quad H(k)\phi_n(k) = \epsilon_n\phi_n(k). \]

- For a closed trajectory in the parameter space, the wave function is:
  \[ \psi(t) = e^{-i\gamma(t)}e^{-i \int_0^t dt' \epsilon_n(k(t'))}\phi_n(k) \]

- The Berry phase and the Berry connection:
  \[ \gamma(t) = \oint d\mathbf{k}\mathcal{A}(\mathbf{k}), \mathcal{A}(\mathbf{k}) = -i\phi_n^\dagger(\mathbf{k})\nabla_{\mathbf{k}}\phi_n(\mathbf{k}). \]

- The Berry curvature:
  \[ \Omega = \nabla_{\mathbf{k}} \times \mathcal{A}(\mathbf{k}), \Rightarrow \quad \Omega = \pm \frac{\mathbf{k}}{2\hbar k^3} \]

The Berry curvature and its field lines for the Weyl semimetal
Chiral kinetic equation

Boltzmann equation:

\[
\left[1 - \frac{e}{c}(\mathbf{B}_{\lambda} \cdot \Omega_{\lambda})\right] \partial_t f_{\lambda} + \left\{-e\tilde{E}_{\lambda} - \frac{e}{c} [\mathbf{v}_{\pi} \times \mathbf{B}_{\lambda}] + \frac{e^2}{c} (\tilde{E}_{\lambda} \cdot \mathbf{B}_{\lambda}) \Omega_{\lambda}\right\} \cdot \partial_{\mathbf{p}} f_{\lambda} \\
+ \left\{\mathbf{v}_{\pi} - e \left[\tilde{E}_{\lambda} \times \Omega_{\lambda}\right] - \frac{e}{c} (\mathbf{v}_{\pi} \cdot \Omega_{\lambda}) \mathbf{B}_{\lambda}\right\} \cdot \nabla f_{\lambda} = I_{\text{coll}}(f_{\lambda}),
\]

where \(\tilde{E}_{\lambda} = \mathbf{E}_{\lambda} + (1/e) \nabla \epsilon_{\pi}, \mathbf{v}_{\pi} = \partial_{\mathbf{p}} \epsilon_{\pi}, \epsilon_{\pi} = v_F p \left[1 + \frac{e}{c} (\mathbf{B}_{\lambda} \cdot \Omega_{\lambda})\right].\)

Collision integral:

\[I \propto -\frac{f_{\lambda} - f^{(0)}_{\lambda}}{\tau}.\]

Distribution function:

\[
f_{\lambda} = \frac{1}{e^{[\epsilon_{\pi} - (u \cdot \mathbf{p}) - \lambda \hbar (p \cdot \omega) - \mu_{\lambda}] / T} + 1}.
\]

\(\Omega_{\lambda} = \lambda \hbar \frac{p}{2p^3}\)

Fluid velocity

Vorticity \(\omega = [\nabla \times \mathbf{u}] / 2\)

Anomalous velocity

[D. Xiao, M.-C. Chang, and Q. Niu, R.M.P. 82, 1959 (2010)]
[D.T. Son and N. Yamamoto, P.R.D 87, 085016 (2013)]
[M.A. Stephanov and Y. Yin, P.R.L. 109, 162001 (2012)]
Euler (Navier-Stokes) equation

- Euler (inviscid) equation for the charged electron liquid:

\[
\frac{1}{v_F} \partial_t \left[ \frac{wu}{v_F} + \sigma^{(\epsilon,B)} B + \frac{\hbar\omega n_5}{2} \right] + \nabla P + \mathcal{O}(\nabla)
\]

\[
= -enE + \frac{1}{c} \left[ B \times \left( enu - \frac{\sigma^{(V)} \omega}{3} \right) \right] + \frac{\sigma^{(B)} u (E \cdot B)}{3v_F^2} + \frac{5c\sigma^{(\epsilon,u)} (E \cdot B) \omega}{v_F}
\]

- Dissipative terms
- Electrostatic and Lorentz forces
- Anomalous terms

where \( w = \epsilon + P \), \( \sigma^{(B)} \propto \mu_5 \), \( \sigma^{(\epsilon,u)} = \mathcal{O}(1) \), \( \sigma^{(\epsilon,B)} \propto \mu \mu_5 \).

- Viscosity terms with shear \( \eta \) and bulk \( \zeta \) viscosities:

\[-\eta \Delta u - \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot u) .\]
Energy conservation relation

- Energy conservation equation:

\[
\partial_t \varepsilon + (\nabla \cdot \mathbf{u}) w + O(\nabla) = -E \cdot \left( enu - \sigma^{(B)} \mathbf{B} - \frac{\sigma^{(V)} \omega}{3} \right).
\]

- Viscosity and thermoconductivity terms:

\[
-\eta (u \Delta u) - \left( \zeta + \frac{\eta}{3} \right) (u \cdot \nabla)(\nabla \cdot u) - \kappa \nabla \cdot \left( \nabla T - \frac{T}{w} \nabla P \right).
\]
Electric and chiral currents

- **Currents:**
  \[
  \mathbf{J} \approx -en\mathbf{u} + \sigma\mathbf{E} + \kappa_e \nabla T + \frac{\sigma_5}{e} \nabla \mu_5 + \sigma^{(V)}(\mathbf{\omega}) + \sigma^{(B)}(\mathbf{B}) + \frac{[\nabla \times \omega] \sigma^{(\epsilon,V)}}{2} + \frac{e^2}{2\pi^2 \hbar^2 c} b_0 \mathbf{B} - \frac{e^2}{2\pi^2 \hbar} [\mathbf{b} \times \mathbf{E}],
  \]

- **Continuity relations and Maxwell’s equations:**
  \[
  \partial_t \rho + (\nabla \cdot \mathbf{J}) = 0,
  \]
  \[
  \partial_t \rho_5 + (\nabla \cdot \mathbf{J}_5) = -\frac{e^3 (\mathbf{E} \cdot \mathbf{B})}{2\pi^2 \hbar^2 c},
  \]

  \[
  \varepsilon_e \nabla \cdot \mathbf{E} = 4\pi (\rho + \rho_b),
  \]

  \[
  \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
  \]

  \[
  \nabla \times \mathbf{B} = \mu_m \frac{4\pi}{c} \mathbf{J} + \varepsilon_e \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.
  \]
Hydrodynamic AHE voltage

The step-like dependence of the AHE voltage signifies the interplay of hydrodynamic and topological effects.

\[ U = -\int_0^{L_y} E_y(y) \, dy = U_{\text{hydro}} + U_{CS}, \]

\[ U_{CS} = \frac{L_y}{\sigma} \frac{e^3 b_z E_x}{2\pi^2 \hbar^2 c}. \]

\[ U_{\text{hydro}} = -\frac{e^3 b_z E_x}{2\pi^2 \hbar^2 c} \frac{e^2 n^2}{N \sigma^2} \times \left[ L_y - \frac{2}{\lambda_y} \tanh \left( \frac{\lambda_y L_y}{2} \right) \right]. \]
Spatial asymmetry is the characteristic feature of the Chern-Simons terms in the nonlocal transport.
Summary

1. The **hydrodynamic regime** is possible for charge carriers in solids under certain experimentally realizable conditions.

2. Among the most interesting **hydrodynamic phenomena in solids** are the formation of vortices, the negative nonlocal resistance, the Poiseuille-like flow, the breakdown of Matthiessen's rule, etc.

3. **Consistent hydrodynamics** is needed to correctly describe topologically nontrivial chiral media such as Weyl semimetals.

4. The interplay of the Chern-Simons terms and hydrodynamic effects is manifested in the **hydrodynamic AHE**.

5. Weyl nodes separation can be also manifested in the **spatial asymmetry of the electron flow**.