The challenge of discovering QCD critical point

M. Stephanov
Outline

1. Introduction.
   - Critical point. History.
   - QCD Critical point
   - Heavy-Ion Collisions

2. Equilibrium physics of the QCD critical point
   - Critical fluctuations
   - Intriguing data from RHIC BES I

3. Non-equilibrium physics of the QCD critical point (work in progress)
   - Hydrodynamics and fluctuations
   - Hydro+
   - General formalism

4. Summary and Outlook
Cagniard de la Tour (1822): discovered continuous transition from liquid to vapour by heating alcohol, water, etc. in a gun barrel, glass tubes.
Faraday (1844) – liquefying gases:

“Cagniard de la Tour made an experiment some years ago which gave me occasion to want a new word.”

Mendeleev (1860) – measured vanishing of liquid-vapour surface tension: “Absolute boiling temperature”.

Andrews (1869) – systematic studies of many substances established continuity of vapour-liquid phases. Coined the name “critical point”.
van der Waals (1879) – in “On the continuity of the gas and liquid state” (PhD thesis) wrote e.o.s. with a critical point.

Smoluchowski, Einstein (1908,1910) – explained critical opalescence.
Landau – classical theory of critical phenomena
Fisher, Kadanoff, Wilson – scaling, full fluctuation theory based on RG.
Critical opalescence

shining laser light through liquid
Critical point
– end of phase coexistence – is a ubiquitous phenomenon

Water:

Is there one in QCD?
Fundamental constituents – quarks and gluons – are (almost) massless. But hadrons (quasiparticles of QCD) are massive.

$$m_{\text{proton}} = \frac{E_{\text{QCD}}}{c^2}$$

This is the origin of almost all of the visible mass in the Universe.

Color charges and color forces are “confined” within hadrons.

High-energy collisions expose color degrees of freedom and high $T$ environment “liberates” color forces (gluons) and color charges.

The resulting new form of matter is Quark-Gluon Plasma.
Is there a CP between QGP and hadron gas phases?

Q1: Can the two phases continuously transform into each other? Yes.

Lattice QCD at $\mu_B = 0$ – a crossover.
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Lattice QCD at $\mu_B = 0$ – a crossover.

QCD in crossover region: no quasiparticles (not hadrons, not quarks/ gluons). Strongly interacting matter (sQGP). More a liquid than a gas.
Is there a CP between QGP and hadron gas phases?

Q2: Is there phase coexistence, i.e., 1st order transition? * Likely.
Is there a CP between QGP and hadron gas phases?

Q2: Is there phase coexistence, i.e., 1st order transition? *Likely.*

Unfortunately, lattice QCD cannot reach beyond $\mu_B \sim 2T$. 

![Diagram of the Phases of QCD](Image)
Is there a CP between QGP and hadron gas phases?

Q2: Is there phase coexistence, i.e., 1st order transition? *Likely.*

Unfortunately, lattice QCD cannot reach beyond $\mu_B \sim 2T$.

But 1st order transition (and thus C.P.) is ubiquitous in models of QCD: NJL, RM, Holography, Strong coupl. Lattice QCD, ...
How can one discover the QCD critical point?

Essentially two approaches to discovering the QCD critical point. Each with its own challenges.

- **Lattice simulations.**
  
  The *sign problem* restricts reliable lattice calculations to $\mu_B = 0$.

  Under different assumptions one can estimate the position of the critical point, assuming it exists, by extrapolation from $\mu = 0$.

- **Heavy-ion collisions. ** *Non-equilibrium.*
Heavy-ion collisions vs the Big Bang

Similarity: expanding and cooling

Difference: One Event vs many events (cosmic variance vs e.b.e. fluctuations)
Similarity: Expansion accompanied by cooling, followed by freezeout.

Difference: tunable parameter $\mu_B$ via $\sqrt{s}$. 
Assumption for the next part of this talk

H.I.C. are sufficiently close to equilibrium that we can study thermodynamics at freezeout $T$ and $\mu_B$ — as a first approximation.
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NB: Event-by-event fluctuations:

Heavy-ion collisions create systems which are large (thermodynamic limit), but not too large ($N \sim 10^2 - 10^4$ particles)

EBE fluctuations are small ($1/\sqrt{N}$), but measurable.
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What are the signatures of the critical point?

EBE fluctuations vs $\sqrt{s}$  

- Equilibrium = maximum entropy.

$$P(\sigma) \sim e^{S(\sigma)} \quad (Einstein \ 1910)$$
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$\delta\sigma \sim \frac{1}{\sqrt{V}}$

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EBE fluctuations vs $\sqrt{s}$

- Equilibrium = maximum entropy.
  
  $$P(\sigma) \sim e^{S(\sigma)} \quad (Einstein \ 1910)$$

- At the critical point $S(\sigma)$ “flattens”. And $\chi \equiv \langle \delta \sigma^2 \rangle V \to \infty$.

$\delta \sigma \sim \frac{1}{\sqrt{V}}$

$\chi \sim V^{2/3}$

CLT?
What are the signatures of the critical point?

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CLT?

$\delta \sigma$ is not an average of $\infty$ many uncorrelated contributions: $\xi \to \infty$

In fact, $\langle \delta \sigma^2 \rangle \sim \xi^2 / V$. 
Higher order cumulants

- $n > 2$ cumulants (shape of $P(\sigma)$) depend stronger on $\xi$.
  
  E.g., $\langle \sigma^2 \rangle \sim \xi^2$ while $\kappa_4 = \langle \sigma^4 \rangle_c \sim \xi^7$ \cite{PRL102(2009)032301}

- For $n > 2$, sign depends on which side of the CP we are.
  
  This dependence is also universal. \cite{PRL107(2011)052301}

- Using Ising model variables:

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![Diagram showing critical point behavior](image)
Equilibrium $\kappa_4$ vs $\mu_B$ and $T$:

In QCD $(t, H) \rightarrow (\mu - \mu_{CP}, T - T_{CP})$

Pradeep-MS 1905.13247
Equilibrium $\kappa_4$ vs $\mu_B$ and $T$:

- In QCD $(t, H) \rightarrow (\mu - \mu_{CP}, T - T_{CP})$

$\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$

Pradeep-MS 1905.13247

1104.1627
Beam Energy Scan I: intriguing hints

Equilibrium $\kappa_4$ vs $\mu_B$ and $T$:
Beam Energy Scan I: intriguing hints

Equilibrium $\kappa_4$ vs $\mu_B$ and $T$:

- Intriguing hint (2015 LRPNS)
- M. Stephanov
Beam Energy Scan I: intriguing hints

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“intriguing hint” (2015 LRPNS)
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Non-equilibrium physics is essential near the critical point.

The challenge taken on by BEST

Goal: build a *quantitative* theoretical framework describing critical point signatures for comparison with experiment.

Strategy:

- Parameterize QCD EOS with yet unknown $T_{CP}$ and $\mu_{CP}$ as variable parameters (e.g., Parotto et al, 1805.05249).
- Use the EOS in a hydrodynamic simulation and compare with experiment to determine or constrain $T_{CP}$ and $\mu_{CP}$. 
Hydrodynamic eqs. are conservation equations ($\partial_\mu T^{\mu\nu} = 0$):

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi];$$
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**Stochastic variables** $\bar{\psi} = (\bar{T}^0, \bar{J}^0)$ are local operators coarse-grained (over “cells” $b$: $\ell_{\text{mic}} \ll b \ll L$):

$$\partial_t \bar{\psi} = -\nabla \cdot \left( \text{Flux}[\bar{\psi}] + \text{Noise} \right)$$

(Landau-Lifshitz)
Stochastic hydrodynamics

Hydrodynamic eqs. are conservation equations ($\partial_{\mu} T^{\mu\nu} = 0$):

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi];$$

Stochastic variables $\tilde{\psi} = (\tilde{T}^{i0}, \tilde{J}^0)$ are local operators coarse-grained (over “cells” $b$: $\ell_{\text{mic}} \ll b \ll L$):

$$\partial_t \tilde{\psi} = -\nabla \cdot \left( \text{Flux}[\tilde{\psi}] + \text{Noise} \right)$$

(Landau-Lifshitz)

Linearized version has been considered and applied to heavy-ion collisions (Kapusta-Muller-MS, Kapusta-Torres-Rincon, . . .

Non-linearities + point-like noise $\Rightarrow$ UV divergences. In numerical simulations – cutoff dependence.
Deterministic approach

Variables are one- and two-point functions:
\[ \psi = \langle \psi \rangle \quad \text{and} \quad G = \langle \psi \bar{\psi} \rangle - \langle \psi \rangle \langle \bar{\psi} \rangle \quad \text{– equal-time correlator} \]

Nonlinearities lead to dependence of flux on \( G \).

\[ \partial_t \psi = -\nabla \cdot \text{Flux}[\psi, G]; \quad \text{(conservation)} \]
\[ \partial_t G = L[G; \psi]. \quad \text{(relaxation)} \]

In Bjorken flow by Akamatsu \textit{et al}, Martinez-Schaefer.
For arbitrary relativistic flow – by An \textit{et al} (this talk).
Earlier, in \textit{nonrelativistic} context, – by Andreev in 1970s.
Deterministic approach

- Variables are one- and two-point functions:
  \[ \psi = \langle \tilde{\psi} \rangle \quad \text{and} \quad G = \langle \tilde{\psi} \tilde{\psi} \rangle - \langle \tilde{\psi} \rangle \langle \tilde{\psi} \rangle \quad \text{– equal-time correlator} \]

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- Advantage: deterministic equations.

  “Infinite noise” causes UV renormalization of EOS and transport coefficients – can be taken care of \textit{analytically} \(^{(1902.09517)}\)
Fluctuation dynamics near CP requires two main ingredients:

- Critical fluctuations \( (\xi \rightarrow \infty) \)
- Slow relaxation mode with \( \tau_{\text{relax}} \sim \xi^3 \) (leading to \( \zeta \rightarrow \infty \))
Fluctuation dynamics near CP requires two main ingredients:

- Critical fluctuations ($\xi \to \infty$)

- Slow relaxation mode with $\tau_{\text{relax}} \sim \xi^3$ (leading to $\zeta \to \infty$)

Both described by the same object: the two-point function of the slowest hydrodynamic mode $m \equiv (s/n)$, i.e., $\langle \delta m(x_1) \delta m(x_2) \rangle$.

Without this mode, hydrodynamics would break down near CP when $\tau_{\text{expansion}} \sim \tau_{\text{relax}} \sim \xi^3$. 
Additional variables in Hydro+

At the CP the *slowest* new variable is the 2-pt function \( \langle \delta m \delta m \rangle \) of the slowest hydro variable:

\[
\phi_Q(x) = \int_{\Delta x} \langle \delta m(x_+) \, \delta m(x_-) \rangle \, e^{iQ \cdot \Delta x}
\]

where \( x = (x_+ + x_-)/2 \) and \( \Delta x = x_+ - x_- \).

Wigner transformed b/c dependence on \( x \) (\( \sim L \)) is slow and relevant \( \Delta x \ll L \). Scale separation similar to kinetic theory.

![Diagram of \( \Delta x \) and \( L \)]
As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_+(\epsilon, n, \phi_Q) = s(\epsilon, n) + \frac{1}{2} \int Q \left( \log \frac{\phi_Q}{\bar{\phi}_Q} - \frac{\phi_Q}{\bar{\phi}_Q} + 1 \right)$$
Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s(\epsilon, n, \phi_Q) = s(\epsilon, n) + \frac{1}{2} \int_Q \left( \log \frac{\phi_Q}{\bar{\phi}_Q} - \frac{\phi_Q}{\bar{\phi}_Q} + 1 \right)$$

- Entropy = log # of states, which depends on the width of $P(m_Q)$, i.e., $\phi_Q$:

  - Wider distribution – more microstates – more entropy: $\log(\phi/\bar{\phi})^{1/2}$ ;

  vs

  - Penalty for larger deviations from peak entropy (at $\delta m = 0$): $-(1/2)\phi/\bar{\phi}$.

  Maximum of $s(\epsilon, n, \phi_Q)$ is achieved at $\phi = \bar{\phi}$. 

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QCD Critical Point

ASU 2020 26 / 36
Hydro+ mode kinetics

The equation for $\phi_Q$ is a relaxation equation:

$$(u \cdot \partial)\phi_Q = -\gamma_\pi(Q)\pi_Q, \quad \pi_Q = -\left(\frac{\partial s_+(\epsilon)}{\partial \phi_Q}\right)_{\epsilon,n}$$

$\gamma_\pi(Q)$ is known from mode-coupling calculation in ‘model H’. It is universal (Kawasaki function).

$\gamma_\pi(Q) \sim 2DQ^2$ for $Q \ll \xi^{-1}$. ($D \sim 1/\xi$).

Characteristic rate: $\Gamma(Q) \sim \gamma_\pi(Q) \sim \xi^{-3}$ at $Q \sim \xi^{-1}$.

Slowness of this relaxation process is behind the divergence of $\zeta \sim 1/\Gamma \sim \xi^3$ and the breakdown of ordinary hydro near CP (frequency dependence of $\zeta$ at $\omega \sim \xi^{-3}$).
Towards a general deterministic formalism

To embed Hydro+ into a unified theory for critical as well as non-critical fluctuations we develop a general *deterministic* (hydro-kinetic) formalism.

We expand hydrodynamic eqs. in \( \{ \delta m, \delta p, \delta u^\mu \} \sim \phi \) and then average, using equal-time correlator

\[
G(x, y) \equiv \langle \phi(x + y/2) \phi(x - y/2) \rangle.
\]

What is “equal-time” in *relativistic* hydro?

Renormalization.
Equal time

We use equal-time correlator \( G = \langle \phi(t, x_+) \phi(t, x_-) \rangle \).


The most natural choice is local \( u(x) \) (\( x = (x_+ + x_-)/2 \)).

Derivatives wrt \( x \) at “\( y \)-fixed” should take this into account:

\[
\Delta x \cdot \bar{\nabla} G(x, y) \equiv G(x + \Delta x, \Lambda(\Delta x)^{-1}y) - G(x, y).
\]

not \( G(x + \Delta x, y) - G(x, y) \).
Confluent derivative, connection and correlator

Take out dependence of components of $\phi$ due to change of $u(x)$:

$$\Delta x \cdot \vec{\nabla} \phi = \Lambda(\Delta x) \phi(x + \Delta x) - \phi(x)$$

Confluent two-point correlator:

$$\tilde{G}(x, y) = \Lambda(y/2) \langle \phi(x + y/2) \phi(x - y/2) \rangle \Lambda(-y/2)^T$$

(boost to $u(x)$ – rest frame at midpoint)

$$\bar{\nabla}_\mu \tilde{G}_{AB} = \partial_\mu \tilde{G}_{AB} - \bar{\omega}^C_{\mu A} \tilde{G}_{CB} - \bar{\omega}^C_{\mu B} \tilde{G}_{AC} - \bar{\omega}^b_{\mu a} y^a \frac{\partial}{\partial y^b} \tilde{G}_{AB}.$$ 

Connection $\bar{\omega}$ corresponds to the boost $\Lambda$.

Connection $\hat{\omega}$ makes sure derivative is independent of the choice of local space triad $e_a$ needed to express $y \equiv x_+ - x_-$. 

We then define the Wigner transform $W_{AB}(x, q)$ of $\tilde{G}_{AB}(x, y)$. 

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QCD Critical Point

ASU 2020
Sound-sound correlation and phonon kinetic equation

Upon lots of algebra with many *miraculous* cancellations we arrive at “hydro-kinetic” equations for components of $W$. The longitudinal components, corresponding to $p$ and $u^\mu$ fluctuations at $\delta(s/n) = 0$, obey the following eq. ($N_L \equiv W_L/(wc_s|q|)$)

\[
\left[(u + v) \cdot \bar{\nabla} + f \cdot \frac{\partial}{\partial q}\right] N_L = -\gamma L q^2 \left(N_L - \frac{T}{c_s|q|} \right) - \gamma L q^2 \left(N_L^{(0)} - \frac{T}{c_s|q|} \right)
\]

$\mathcal{L}[N_L]$ – Liouville op.

Kinetic eq. for phonons with $E = c_s|q|$, $v = c_s q/|q|$ ($q \cdot u = 0$)

\[
f_\mu = -E(a_\mu + 2v^{\nu}\omega_{\nu\mu}) - q^{\nu} \partial_{\perp \mu} u_\nu - \bar{\nabla}_{\perp \mu} E.
\]

inertial + Coriolis

“Hubble”

$N_L^{(0)}$ is equilibrium Bose-distribution.
Diffusive mode fluctuations

Fluctuations of \( m \equiv s/n \) and transverse components of \( u^\mu \) obey

(entropy-entropy) \[ \mathcal{L}[N_{mm}] = -2 \Gamma_\lambda \left( N_{mm} - \frac{c_p}{n} \right) + \ldots \]

(entropy-velocity) \[ \mathcal{L}[N_{mi}] = -2(\Gamma_\eta + \Gamma_\lambda)N_{mi} + \ldots \]

(velocity-velocity) \[ \mathcal{L}[N_{ij}] = -2 \Gamma_\eta \left( N_{ij} - \frac{Tw}{n} \right) + \ldots \]

\( \mathcal{L} \) is Liouville operator with \( v = f = 0 \), i.e., no propagation, but diffusion: \( \Gamma_X = \gamma_X q^2 \), where \( \gamma_\lambda = \lambda/c_p \) and \( \gamma_\eta = \eta/w \).

“...” are terms \( \sim \) background grads, mixing \( N_{mm} \leftrightarrow N_{mi} \leftrightarrow N_{ij} \).

Near critical point \( \Gamma_\lambda \) is smallest, \( \gamma_\lambda = \lambda/c_p \sim 1/\xi \rightarrow 0 \).

\( N_{mm} \) equation decouples and matches Hydro+ \( (\phi_Q = nN_{mm}) \). Very nontrivially!
Beyond Hydro+

- Hydro+ breaks down when hydro frequency/rate is of order $\xi^{-2}$ due to next-to-slowest modes ($N_{mi}$ and $N_{ij}$).

- The formalism extends Hydro+ to higher frequencies, i.e., shorter hydrodynamic scales (all the way to $\xi$).

Fluctuations ($N_{mi}$) enhance conductivity for small $\omega$. 
Renormalization

Expansion of $\langle T^{\mu\nu} \rangle$ contains $\langle \phi(x) \phi(x) \rangle = G(x, 0) = \int \frac{d^3 q}{(2\pi)^3} W(x, q)$.

This integral is divergent (equilibrium $G^{(0)}(x, y) \sim \delta^3(y)$).
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$$W(x, q) \sim \begin{cases} W^{(0)} & \text{("OPE" or gradient expansion)} \\
T_w & \\
\partial u/q^2 & \end{cases} + \tilde{W}$$
Renormalization

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\[ W(x, q) \sim W^{(0)}_{Tw} + W^{(1)}_{\partial u/q^2} + \tilde{W} \]

($\sim$“OPE” or gradient expansion)

\[ G(x, 0) \sim \Lambda^3_{\text{ideal (EOS)}} + \Lambda \partial u_{\text{visc. terms}} + \tilde{G}_{\text{finite \ "\partial^{3/2}"}} \]
Renormalization

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W(x, q) \sim W^{(0)} + W^{(1)} + \tilde{W}
$$

($\sim$“OPE” or gradient expansion)

$$
G(x, 0) \sim \Lambda^3 + \Lambda \partial u + \tilde{G}
$$

ideal (EOS) visc. terms finite $\partial^{3/2}$

$$
\langle T^{\mu\nu}(x) \rangle = \epsilon u^\mu u^\nu + p(\epsilon, n)\Delta^{\mu\nu} + \Pi^{\mu\nu} + \left\{ G(x, 0) \right\}
$$

$$
= \epsilon_R u^\mu_R u^\nu + p_R(\epsilon_R, n_R)\Delta_R^{\mu\nu} + \Pi_R^{\mu\nu} + \left\{ \tilde{G}(x, 0) \right\}.
$$
Work in progress and outlook

- Add higher-order correlators for non-gaussian fluctuations.
- Connect fluctuating hydro with freezeout kinetics and implement in full hydrodynamic code and event generator. Compare with experiment.
- First-order transition in fluctuating hydrodynamics?
- Connection to action principle (SK) formulation.
A fundamental question about QCD phase diagram:

*Is there a critical point on the QGP-HG boundary?*

Intriguing results from experiments (BES-I).
More to come from BES-II (also FAIR/CBM, NICA, J-PARC).

Quantitative theoretical framework is needed ⇒ BEST COLLABORATION.

In H.I.C., the magnitude of the fluctuation signatures of CP is controlled by dynamical *non-equilibrium effects*.

In turn, critical fluctuations affect hydrodynamics.

The interplay of critical and dynamical phenomena: Hydro+. 