Hydrodynamization and attractors in rapidly expanding fluids

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Special Theoretical Physics Seminar
Far-from-equilibrium

Equilibrium

?
Today: Attractors in kinetic theory and fluid dynamics out of equilibrium
Far-from-equilibrium

Hydrodynamics
Hydrodynamics: one theory to rule them all

Quark-Gluon Plasma

New discoveries: Nearly Perfect Fluids

\[ T \sim 10^{12} \, K \]

Ultracold atoms

\[ T \sim 10^{-7} \, K \]
 Fluidity in Heavy Ions

$\frac{dN}{d\phi} = \frac{N}{2\pi} \left( 1 + \sum_n (2v_n \cos(n\phi)) \right)$

$v_n$ provides information of the initial spatial geometry of the collision
Fluidity in Cold Atoms

Aspect ratio measures pressures anisotropies

Cao et. al (2010)
Size of the hydrodynamical gradients

Heavy Ion Collisions

Martinez et. al. (2012)

\[ \frac{P_L}{P_T} \text{ at } \tau = 2.50 \text{ fm/c} \]

Pressure anisotropies are not small

Cold Atoms

Paradox:
Hydrodynamics provides a good description despite large gradients…. Why?

Introductory textbook: Hydrodynamics works as far as there is a hierarchy of scales

\[
Kn = \frac{l_{micro}}{L_{macro}} \ll 1
\]
Hydro as an effective theory

Coarse-grained procedure reduces # of degrees of freedom

Microscopic: $10^{23}$ particles
Mesoscopic: $10^7 - 10^9$ particles
Continuum: $T, \mu, \mu_i, \epsilon, n, p, ...$
Hydro as an effective theory

How does hydrodynamical limit emerges from an underlying microscopic theory?

Microscopic: $10^{23}$ particles

Mesoscopic: $10^7 - 10^9$ particles

Continuum: $T, \mu, \mu_i, \epsilon, n, p, \ldots$
Kinetic theory: Boltzmann equation

Microscopic dynamics is encoded in the distribution function $f(t, x, p)$.

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = -C[f]$$

Diffusion  External Force  Particle imbalance

$C[f] =$

Gain  Lose
Asymptotics in the Boltzmann equation

Usually the distribution function is expanded as series in Kn, i.e.,

\[ f(x^\mu, p) = \sum_{k=0}^{\infty} (Kn)^k f_k(x^\mu, p) \]

Macroscopic quantities are simply averages, e.g.,

\[ T^{\mu\nu} = \int p^\mu p^\nu f(x^\mu, p) \quad \rightarrow \quad T^{\mu\nu} = \sum_{k=0}^{\infty} (Kn)^k T_k^{\mu\nu} \]

\[ T_0^{\mu\nu} = (\epsilon + p(\epsilon)) u^\mu u^\nu + p(\epsilon) g^{\mu\nu} \quad \rightarrow \quad \mathcal{O}(Kn^0): \text{Ideal fluid} \]

\[ T_1^{\mu\nu} = -\eta \sigma^{\mu\nu} \quad \rightarrow \quad \mathcal{O}(Kn): \text{Navier-Stokes} \]

\[ T_2^{\mu\nu} \quad \rightarrow \quad \mathcal{O}(Kn^2): \text{IS, etc} \]
Warning

\[ Kn \sim \frac{l}{L} \sim \lambda_{mfp} \nabla \cdot \vec{v} \sim \mathcal{O}(1) \]
Attractor in hydrodynamics

- Different IC
- NS
- IS
- Attractor

Same late time behavior independent of the IC!!!

Heller and Spalinski (2015)

\[ f = \frac{2}{3} + \frac{\pi}{4\epsilon} \]

\[ \omega = \tau T \quad \text{[Kn]}^{-1} \]
Divergence of the late-time perturbative expansion

Heller & Spalinski:

\[ \bar{\pi} = \sum_{k=1}^{\infty} a_k [Kn]^k \]

- Far from equilibrium
- Close to equilibrium
- Large anisotropies \( Kn \sim 1 \)
- \( -0.75 \)

\[ w = \tau T(\tau) \]

\[ [Kn]^{-1} \]
Divergence of perturbative series

\[ \bar{\pi} = \sum_{k=1}^{\infty} a_k [Kn]^k \]

\[ \lim_{k \to \infty} a_k \approx k! \]

Perturbative asymptotic expansion is divergent!!!!

Heller and Spalinski (2015)
Resurgence and transseries

Asymptotic expansion

\[ \bar{\pi} = \sum_{k=1}^{\infty} a_k \left[ \text{Kn} \right]^k \]

Transseries solutions

\[ \bar{\pi} = \sum_{k=1}^{\infty} a_k + \sum_{l=1}^{\infty} u_{k,l} \left( \sigma e^{-S/\text{Kn}} \left[ \text{Kn} \right]^\beta \right)^l \left[ \text{Kn} \right]^k \]

Costin (1998)

Non-perturbative

'Instanton' Non-hydro modes

Perturbative

\[ \bar{\pi} = \frac{2}{\beta} (P_L - P_T) / \epsilon \]

\[ w = \tau T(\tau) \]

\[ \left[ \text{Kn} \right]^{-1} \]
Message to take I

Romatschke (2017)

- Arbitrarily far-from-equilibrium initial conditions used to solve hydro equations merge towards a unique line (attractor).
- Independent of the coupling regime.
- Attractors can be determined from very few terms of the gradient expansion.
- At the time when hydrodynamical gradient expansion merges to the attractor, the system is far-from-equilibrium, i.e. large pressure anisotropies are present in the system $P_L \neq P_T$
Message to take I

Existence of a new theory for far-from-equilibrium fluids

- What are their properties?
Do we have experimental evidence?

Nagle, Zajc (2018)

Flow-like behavior has been measured in collisions of small systems

- Hydrodynamical models seem to work in p-Au and d-Au collisions
Physical meaning:
Transient non-newtonian behavior

\[
\bar{\pi} = \sum_{k=1}^{\infty} [Kn]^k \left[ a_k + \sum_{l=1}^{\infty} u_{k,l} \left( \sigma e^{-S/\text{Kn}} [Kn]^\beta \right)^l \right] \]

\[ F_k(\sigma e^{-S/\text{Kn}} [Kn]^\beta) \]

Each function \( F_k \) satisfies:

\[ \lim_{Kn \to 0} F_k = a_k \]

\[ \frac{F_k}{d(Kn^{-1})} = \beta_k(Kn, F_k, F_{k+1}, F_{k+2}, \ldots) \]

Dynamical RG flow structure!!!
Physical meaning: Transient non-newtonian behavior

- Generalizes the concept of transport coefficient for far-from-equilibrium!!!
- It depends on the story of the fluid and thus, its rheology
- It presents shear thinning and shear thickening

\[ \frac{\eta}{s} = -\frac{3}{40} F_{1,1}(w) \]
Non-hydrodynamic transport

Hydro vs. Non-hydro modes

Hydro breaks down around $p_T \sim 2.5$ GeV
Non-hydro modes are dominant at $p_T \geq 2.5$ GeV
Non-hydrodynamic transport

Breaking of hydrodynamics

\[ E_p = 2.5 \text{ GeV} \]

\[ \delta f \] measures deviations from equilibrium of the full distribution function

Including only one mode (hydro)

\[ \delta f_s \sim a \bar{\pi} \bar{\pi} \]

Including two modes (non-hydro)

\[ \delta f_{s+sh} \sim a \bar{\pi} \bar{\pi} + a_{\bar{c}_{sh}} \bar{c}_{sh} \]

Martinez et. al., (2018, 2019)
Non-hydrodynamic transport

For intermediate scales of momentum $\delta f(t,x,p)$ requires the two slowest non-hydro modes in the soft and semi-hard momentum sectors.

Non-hydrodynamic transport: dynamics of non-hydro modes and hydro modes

⇒ Cold atoms: pressure anisotropies as non-hydrodynamic degrees of freedom (Bluhm & Schaefer, 2015-2017)
Non-hydrodynamic transport

For intermediate scales of momentum $\delta f(t, x, p)$ requires the two slowest non-hydro modes in the soft and semi-hard momentum sectors.

Non-hydrodynamic transport: dynamics of non-hydro modes and hydro modes.

The asymptotic late time attractor of the distribution function depends not only on the shear but also on other slowest non-hydro modes!!!
Attractors in higher dimensions: Gubser flow for IS theory

A. Behtash, CN Cruz, M. Martinez

arXiv: PRD in press
Attractors in higher dimensions: Gubser flow for IS theory

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Late time asymptotic attractor

No universal line during intermediate stages
Attractors in higher dimensions: Gubser flow for IS theory

A. Behtash, CN Cruz, M. Martinez
arXiv: PRD in press

Attractor is a 1-d non planar manifold

- In Bjorken you see a unique line cause the attractor is a 1d planar curve
Attractors in higher dimensions: Gubser flow for IS theory

- Asymptotic behavior of temperature is not determined by the Knudsen number
- Breaking of asymptotic gradient expansion (see also Denicol & Noronha)
Research directions and opportunities

- Emergence of liquid-like behavior in systems at extreme conditions
  Neutron star mergers, cosmology, chiral effects in nuclear and condensed matter systems
- Early time behavior of attractors
  Behtash et. al., Wiedemann et. al., Heinz et. al.
- Entropy production & experiments
  Giacalone et. al.
- Higher dimensional attractors via machine learning
  Heller et. al.
- Understanding scaling behavior
  Mazeliauskas and Berges, Venugopalan et. al., Gelis & others
Conclusions

- Hydrodynamics is a beautiful 200 year old theory which remains as one of the most active research subjects in physics, chemistry, biology, etc.

- The emergence of liquid-like behavior has been observed in a large variety of systems subject to extreme conditions

- We need new ideas to formulate an universal Fluid dynamics for equilibrium and non-equilibrium

- Need to test these ideas with experiments
Backup slides
Comparing Gubser flow attractors

Anisotropic hydrodynamics matches the exact attractor to higher numerical accuracy !!!

Anisotropic hydro is an effective theory which resumes the largest anisotropies of the system in the leading order term
Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

\[ SO(3)_q \otimes SO(1, 1) \otimes Z_2 \]

- Special Conformal transformations + rotation along the beam line
- Boost invariance
- Reflections along the beam line
Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

\[ SO(3)_q \otimes SO(1, 1) \otimes Z_2 \]

In polar Milne Coordinates \((\tau, r, \phi, \eta)\)

\[ u^\mu = (\cosh \kappa(\tau, r), \sinh \kappa(\tau, r), 0, 0) \]

\[ \kappa(\tau, r) = \tanh^{-1} \left( \frac{2q^2 \tau r}{1 + (qr)^2 + (q\tau)^2} \right) \]

\( q \) is a scale parameter
Gubser flow

\[ g_{\mu\nu}(x) \rightarrow e^{-2\Omega(x)}g_{\mu\nu}(x) \]

Flat Minkowski space

\[
\sinh \rho = -\frac{1 - \tau^2 + r^2}{2\tau}, \quad \tan \theta = \frac{2\tau}{1 + \tau^2 - r^2}.
\]

Complicated dynamics

\[
x^\mu = (\tau, r, \phi, \eta) \quad \Rightarrow \quad \hat{x}^\mu = (\rho, \theta, \phi, \eta)
\]

\[
ds^2 = -d\tau^2 + dr^2 + r^2 d\phi^2 + d\eta^2 \quad \Rightarrow \quad d\hat{s}^2 = -d\rho^2 + \cosh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) + d\eta^2
\]

\[
u^\mu = (u^\tau(\tau, r), u^\tau(\tau, r), 0, 0) \quad \Rightarrow \quad \hat{u}^\mu = (1, 0, 0, 0)
\]

\[
\epsilon(\tau, r) \quad \Rightarrow \quad \hat{\epsilon}(\rho)
\]

3d de Sitter space
Exact Gubser solution

- In $dS \otimes R$ the dependence of the distribution function is restricted by the symmetries of the Gubser flow
  \[ f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \hat{p}_\Omega^2, \hat{p}_\eta) \]
  \[ \hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2 \theta} \]
  Total momentum in the $(\theta, \phi)$ plane
  \[ \hat{p}_\eta \]
  Momentum along the $\eta$ direction

- The RTA Boltzmann equation gets reduced to
  \[ \frac{\partial}{\partial \rho} f(\rho, \hat{p}_\Omega^2, \hat{p}_\eta) = -\frac{\hat{T}(\rho)}{c} \left( f(\rho, \hat{p}_\Omega^2, \hat{p}_\eta) - f_{eq} \left( \frac{\hat{p}^2}{\hat{T}(\rho)} \right) \right) \]
  \[ c = 5 \frac{\eta}{S} \]

- The exact solution to this equation is
  \[ f(\rho, \hat{p}_\Omega^2, \hat{p}_\eta) = D(\rho, \rho_0) f_0(\rho, \hat{p}_\Omega^2, \hat{p}_\eta) + \frac{1}{c} \int_{\rho_0}^{\rho} d\rho' \hat{T}(\rho') D(\rho, \rho') f_{eq} \left( \frac{\hat{p}^2}{\hat{T}(\rho)} \right) \]
Boltzmann equation

The macroscopic quantities of the system are simply averages weighted by the solution for the distribution function

\[
\varepsilon(x) = \int \frac{d^3 p}{\sqrt{-g p^0}} (p \cdot u)^2 f(x^\mu, p_i),
\]

\[
\mathcal{P}(x) = \frac{1}{3} \int \frac{d^3 p}{\sqrt{-g p^0}} \Delta_{\mu\nu} p^\nu p^\mu f(x^\mu, p_i),
\]

\[
\pi^{\mu\nu}(x) = \int \frac{d^3 p}{\sqrt{-g p^0}} p^{(\mu} p^{\nu)} f(x^\mu, p_i).
\]

Solving exactly the Boltzmann eqn. is extremely hard so one needs some method to construct approximate solutions.
Fluid models for the Gubser flow

\[ \frac{\partial \rho \hat{T}}{\hat{T}} + \frac{2}{3} \tanh \rho = \frac{\bar{\pi}}{3} \tanh \rho \]

\[ \hat{\tau}_{\bar{\pi}} \left( \partial_\rho \bar{\pi} + \frac{4}{3} (\bar{\pi})^2 \tanh \rho \right) + \bar{\pi} = \frac{4}{\hat{T}} \frac{\eta}{3} \tanh \rho + \frac{10}{7} \hat{\tau}_{\bar{\pi}} \bar{\pi} \tanh \rho \]

\[ \partial_\rho \bar{\pi} + \frac{\bar{\pi}}{\hat{\tau}_r} = \frac{4}{3} \tanh \rho \left( \frac{5}{16} + \bar{\pi} - \bar{\pi}^2 - \frac{9}{16} \mathcal{F}(\bar{\pi}) \right) \]
In the Gaussian approximation (white random noise)

\[
G_{SO}^{\omega_0}(\omega, 0) = \int \frac{d\omega'}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[ 2a_{\rho\rho}^2 \Delta_{S}^{\rho\rho}(\omega', k) \Delta_{S}^{\rho\rho}(\omega - \omega', k) + a_{\rho T}^2 \Delta_{S}^{\rho\rho}(\omega', k) \Delta_{S}^{TT}(\omega - \omega', k) + 2a_{TT}^2 \Delta_{S}^{TT}(\omega', k) \Delta_{S}^{TT}(\omega - \omega', k) \right].
\]

\[
\Delta_{S}^{TT}(\omega, k) = \frac{2T^2}{c_P} \frac{D_T k^2}{\omega^2 + (D_T k^2)^2}
\]

\[
\Delta_{S}^{\rho\rho}(\omega, k) = 2\rho T \left\{ \frac{\Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} + \frac{\Delta c_P}{c_s^2} \frac{D_T k^2}{\omega^2 + (D_T k^2)^2} + \frac{\Delta c_P}{c_s^2} \frac{(\omega^2 - c_s^2 k^2) D_T k^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} \right\}
\]

Dominated by the diffusive heat wave

Mix of sound and diffusive modes
Statistical field theory method

After a long algebra plus pole analysis of propagators

\[
G_{R \bar{R}}^{\omega \omega}(\omega, 0) = -A_T L(\omega, \Lambda, 2D_T) - A_\Gamma L(\omega, \Lambda, \Gamma)
\]

\[
L(\omega, \Lambda, D_i) = \frac{1}{2\pi^2} \left\{ \frac{\Lambda^3}{3} + \frac{i\omega \Lambda}{D_i} - \frac{\pi}{2\sqrt{2}} (1 + i) \left( \frac{\omega}{D_i} \right)^{3/2} + \ldots \right\}.
\]
Resurgence and transseries

A new time-dependent resummation scheme is needed

![Math equation: \[ \bar{\pi} = \sum_{k=1}^{\infty} a_k \left[ Kn \right]^k \]]

Asymptotic expansion

Resurgence
Costin (1998)

![Math equation: \[ \bar{\pi} = \sum_{k=1}^{\infty} \left[ a_k + \sum_{l=1}^{\infty} u_{k,l} \left( \sigma e^{-S/ Kn} \left[ Kn \right]^\beta \right)^l \right] \left[ Kn \right]^k \]]

Transseries solution

Transseries:

At a given order of the perturbative expansion, transseries resumes the non-p perturbative contributions of small perturbations around the asymptotic late time fixed point
Size of the hydrodynamical gradients

Heavy Ion Collision

Martinez et. al. (2012)

\[ \frac{P_L}{P_T} \text{ at } \tau = 2.50 \text{ fm/c} \]

\[ y \text{ [fm]} \]

\[ x \text{ [fm]} \]

LARGE UNCERTAINTY

Bass et. al. (2017)

\[ \frac{\eta}{s} \]

\[ \text{Temperature [GeV]} \]

KSS bound \( \frac{1}{4\pi} \)

Cold Atoms

Gradients are not small

\[ Kn \sim 1 \]

Large uncertainty

O’Hara et. al. (2002)

Schaefer (2007)
Universality of hydrodynamics

- Fluid dynamical equations of motion are **universal**

  ⇒ **In general** fluid dynamics is **not** a particular limit of a weakly (e.g. kinetic theory) or strongly coupled (e.g. AdS/CFT) theory

- **Transport coefficients** (e.g. shear viscosity) and other thermodynamical properties depend on microscopic details of the system

- Hydrodynamical approach also describes **heat conduction, volume expansion,** etc.

\[
\Pi = -\zeta \nabla \cdot \vec{v}
\]

\[
\vec{q} = -\kappa \nabla T
\]
Non-newtonian fluids and rheology
Non-newtonian fluids and rheology

Shear viscosity

- Becomes a function of the gradient of the flow velocity
- can increase (shear thickening) or decrease (shear thinning) depending on the size of the gradient of the flow velocity

\[ \pi_{yx} \sim \eta \partial_y v_x \]

\[ \pi_{yx} \sim \eta (\partial_y v_x) \partial_y v_x \]
Non-newtonian fluids and rheology

\[ \pi_{yx} \sim \eta \partial_y v_x \]

\[ \pi_{yx} \sim \eta (\partial_y v_x) \partial_y v_x \]

Does the QGP behave like a non-newtonian fluid?
Our idea

- Develop a new truncation scheme which captures some of the main features of far-from-equilibrium fluids (e.g. non-hydrodynamical modes) while being simple enough to perform concrete calculations.

\[ \tau_\pi D_\gamma \pi^{\mu\nu} + \pi^{\mu\nu} = \eta \sigma^{\mu\nu} \]

- Keep track of the deformation history of the fluid.

\[ \tau_\pi (\sigma^{\mu\nu}) D_\gamma \pi^{\mu\nu} + \pi^{\mu\nu} = \eta (\sigma^{\mu\nu}) \sigma^{\mu\nu} \]

- Study its rheological properties.

⇒ Study its rheological properties.
Effective $\eta/s$ as a non-hydrodynamical series

At $O(w^{-1})$ the dominant term of the trans-series is

$$c_1 = \sum_l U_{1l}^{-1} \tilde{u}_{l,1}^{(0)} \frac{1}{w}$$

On the other hand, Chapman-Enskog expansion gives the asymptotic behavior of $c_1$

$$c_1 = -\frac{40}{3} \frac{1}{w} \left( \frac{\eta}{s} \right) \frac{1}{0}$$

$$\left( \frac{\eta}{s} \right) = -\frac{3}{40} \sum_l U_{1l}^{-1} \tilde{u}_{l,1}^{(0)}$$

- Effective $\eta/s$ is the asymptotic limit of a trans-series
- We can study its rheology by following the ‘history’ of the corresponding trans-series
Effective $\eta/s$ as a non-hydrodynamic series

Thus effective $\eta/s$ is

$$\left( \frac{\eta}{s} \right)_R = -\frac{3}{40} \sum_l U_{1l}^{-1} \tilde{C}_{l,1}(\sigma e^{-Sw} w^b)$$

Its RG flow evolution is one of the differential recursive relation of the corresponding trans-series

$$\frac{d}{dw} \left( \frac{\eta}{s} \right)_R = -\frac{3}{40} \sum_l U_{1l}^{-1} \frac{d}{dw} \tilde{C}_{l,1}(\sigma e^{-Sw} w^b)$$

Late-time asymptotic value

Non-hydrodynamic mode
Decay determined by Lyapunov exponent