Bulk viscosity for high amplitude oscillations

Prof. Mark Alford
Washington University in St. Louis

M. Alford, S. Mahmoodifar,
K. Schwenger, work in progress.
But there are also non-uniform phases, such as the crystalline ("LOFF"/"FFLO") phase. (Alford, Bowers, Rajagopal, hep-ph/0008208)
Signatures of color superconductivity in compact stars


Gaps in quark spectra and Goldstone bosons affect Transport properties:
emissivity, heat capacity, viscosity (shear, bulk), conductivity (electrical, thermal)...

1. Gravitational waves: r-mode instability, shear and bulk viscosity
2. Glitches and crystalline ("LOFF") pairing
3. Cooling by neutrino emission, neutrino pulse at birth
An r-mode is a quadrupole flow that emits gravitational radiation. It becomes unstable (i.e. arises spontaneously) when a star spins fast enough, and if the shear and bulk viscosity are low enough.

The unstable r-mode can spin the star down very quickly, in days to years (Andersson gr-qc/9706075; Friedman and Morsink gr-qc/9706073; Lindblom astro-ph/0101136).

So if we see a star spinning quickly, we can infer that the interior viscosity must be high enough to damp the r-modes.
Constraints from r-modes: current results

Regions above curves are forbidden ⇐ viscosity is too low to damp r-modes.

But r-modes grow exponentially, so large-amplitude effects must be included.
What is bulk viscosity?

Energy consumed in a compression cycle:

\[ V(t) = \bar{V} + \text{Re}[\delta V \exp(i \omega t)] \]
\[ p(t) = \bar{p} + \text{Re}[\delta p \exp(i \omega t)] \]

\[
\left\langle \frac{dE}{dt} \right\rangle = -\frac{\zeta}{\tau} \int_0^\tau (\text{div} \, \vec{v})^2 dt = \frac{\zeta}{2} \frac{\omega^2}{\bar{V}^2} \frac{\delta V^2}{\bar{V}^2} = -\frac{1}{\tau \bar{V}} \int_0^\tau p(t) \frac{dV}{dt} dt
\]

Physically, bulk viscosity arises from re-equilibration processes. If some quantity goes out of equilibrium on compression, and re-equilibrates on a timescale comparable to \( \tau \), then pressure gets out of phase with volume and energy is consumed. (Just like \( V \) and \( Q \) in an \( R-C \) circuit.)
Calculating bulk viscosity

- Compression at freq $\omega$, so conserved charges oscillate as
  $$n(t) = n_{\text{avg}} + \Delta n \sin(\omega t)$$

- One quantity "a" goes out of equilibrium (e.g., $S - D$ in quark matter). In equilibrium, $\mu_a = 0$.

- EoS is characterized by susceptibilities $B, C$.

$$\zeta = -\frac{1}{\pi} \frac{\bar{n}}{\Delta n} \frac{C}{B} \int_0^\tau \mu_a(t) \cos(\omega t) dt$$

Bulk visc arises from component of $\mu_a$ that lags behind the forcing oscillation by a phase of $90^\circ$; $\mu_a(t)$ is given by

$$\frac{d\mu_a}{dt} = C\omega \frac{\Delta n}{n_{\text{avg}}} \cos(\omega t) - \Gamma(\mu_a, T)$$

Re-express this in dimensionless variables:
Computing departure from equilibrium

- Define dimensionless time (i.e. phase) \( \varphi = \omega t \)
- Define dimensionless departure from equilibrium \( \bar{\mu}_a = \mu_a / T \)
- Driving coeff \( d = \frac{C \Delta n}{T n_{\text{avg}}} \)
- Equilibration rate: \( \Gamma(\mu_a, T) = \tilde{\Gamma} T^\kappa \gamma(\mu_a / T) \).
  
  Equilibration coeff \( f = \frac{B}{\omega} \tilde{\Gamma} T^\kappa \).

\[
\frac{d \bar{\mu}_a}{d \varphi} = d \cos(\varphi) - f \gamma(\bar{\mu}_a)
\]

Dependence on density, EoS, driving amplitude, and temperature is contained in \( d \) and \( f \). Dependence of equilibration rate on \( \bar{\mu}_a \) is contained in \( \gamma(\bar{\mu}_a) = \bar{\mu}_a + \chi_1 \bar{\mu}_a^3 + \cdots \).


**Suprathermal and subthermal bulk viscosity**

**Subthermal**: assume $\bar{\mu}_a \ll 1$ (i.e. $\mu_a \ll T$), so $\gamma(\bar{\mu}_a) = \bar{\mu}_a$,

$$\frac{d\bar{\mu}_a}{d\phi} = d\cos(\phi) - f\bar{\mu}_a$$

$$\bar{\mu}_a(\phi) = -\frac{f}{1 + f^2} \frac{d}{d\phi} \cos \phi + \frac{d}{1 + f^2} \sin \phi$$

$$\zeta_{\text{sub}} = \frac{C}{B\omega} \frac{f}{1 + f^2} = \frac{C}{B} \frac{\gamma_{\text{eff}}}{\omega^2 + \gamma_{\text{eff}}^2}$$

($\gamma_{\text{eff}} \equiv B\tilde{\Gamma}T^\kappa$)
Suprathermal and subthermal bulk viscosity

**Subthermal**: assume $\bar{\mu}_a \ll 1$ (i.e. $\mu_a \ll T$), so $\gamma(\bar{\mu}_a) = \bar{\mu}_a$,

$$\frac{d\bar{\mu}_a}{d\varphi} = d \cos(\varphi) - f \bar{\mu}_a$$

$$\bar{\mu}_a(\varphi) = -\frac{f d}{1 + f^2} \cos \varphi + \frac{d}{1 + f^2} \sin \varphi$$

$$\zeta_{\text{sub}} = \frac{C^2 f}{B \omega (1 + f^2)} = \frac{C^2}{B} \frac{\gamma_{\text{eff}}}{\omega^2 + \gamma_{\text{eff}}^2} \quad (\gamma_{\text{eff}} \equiv B \tilde{\Gamma} T^\kappa)$$

**Suprathermal**: allow $\bar{\mu}_a \gtrsim 1$ (always assuming $\Delta n \ll n_{\text{avg}}$),

$$\frac{d\bar{\mu}_a}{d\varphi} = d \cos(\varphi) - f \bar{\mu}_a \left(1 + \chi_1 \frac{\bar{\mu}_a^2}{T^2} + \cdots + \chi_N \frac{\bar{\mu}_a^{2N}}{T^{2N}} \right)$$

Now there are nonlinear effects; $\bar{\mu}_a(\varphi)$ may not be harmonic.
The subthermal bulk viscosity

\[ \zeta_{\text{sub}}(\omega, T) = P \frac{\gamma_{\text{eff}}(T)}{\gamma_{\text{eff}}(T)^2 + \omega^2} \]

- \( \zeta_{\text{sub}} \) is independent of driving amplitude.
- Prefactor \( P = \frac{C^2}{B} \) is a combination of susceptibilities.
- \( \gamma_{\text{eff}} \) is the effective rate/particle of the re-equilibration process.

\( \omega \quad \gamma_{\text{eff}} \quad \frac{1}{2}P/\omega \)
The subthermal bulk viscosity

\[ \zeta_{\text{sub}}(\omega, T) = P \frac{\gamma_{\text{eff}}(T)}{\gamma_{\text{eff}}(T)^2 + \omega^2} \]

- \( \zeta_{\text{sub}} \) is independent of driving amplitude.
- Prefactor \( P = \frac{C^2}{B} \) is a combination of susceptibilities.
- \( \gamma_{\text{eff}} \) is the effective rate/particle of the re-equilibration process.
- As \( \gamma_{\text{eff}} \to 0 \), \( \zeta \to 0 \). No equilibration.
- As \( \gamma_{\text{eff}} \to \infty \), \( \zeta \to 0 \). Infinitely fast equilibration.
The subthermal bulk viscosity

$$\zeta_{\text{sub}}(\omega, T) = P \frac{\gamma_{\text{eff}}(T)}{\gamma_{\text{eff}}(T)^2 + \omega^2}$$

- $\zeta_{\text{sub}}$ is independent of driving amplitude.
- Prefactor $P = C^2/B$ is a combination of susceptibilities.
- $\gamma_{\text{eff}}$ is the effective rate/particle of the re-equilibration process.
- As $\gamma_{\text{eff}} \to 0$, $\zeta \to 0$. No equilibration.
- As $\gamma_{\text{eff}} \to \infty$, $\zeta \to 0$. Infinitely fast equilibration.
- In phases where Fermi surface modes dominate equilibration (nuclear, unpaired quark matter, 2SC) $P$ is constant for $T \ll \mu_q$, and subthermal bulk viscosity peaks when $\gamma_{\text{eff}}(T) = \omega$.
- In phases where bosons dominate equilibration (CFL, CFL-K0), $P(T)$ washes out the peak.
The general bulk viscosity

To include the suprathermal regime, we have to solve the diffeq for \( \bar{\mu}_a(\varphi) \) numerically.

For a given form of matter, we can summarize dependence on driving amplitude and temperature in dimensionless function \( I(d, f) \),

\[
\zeta = \frac{C^2}{2\omega B} I(d, f)
\]

This could then be used to calculate damping time of \( r \)-modes.

Nuclear matter, modified Urca
Typical amplitude dependence

Unpaired quark matter: \( \gamma(\bar{\mu}_a) = \bar{\mu}_a + \chi_1 \bar{\mu}_a^3 \), can do analytic approx.

- At low \( T \), bulk visc rises rapidly with amplitude when entering the suprathermal region: could set saturation amplitude for r-modes.
- Maximum value is same for all temperatures/amplitudes.
- If amplitude gets too high, bulk viscosity drops again.
Results for bulk viscosity

$T = 10^6$ K

$T = 10^9$ K

- Bulk visc rises very steeply in suprathermal regime
- Max reached at $\Delta n/n \sim 0.1$; max value indp of temperature
- Suprathermal enhancement is greater at low $T$ and for matter where $\zeta$ goes as higher power of $\mu_a$. 
Future directions

Transport:

- Extend to other phases of quark matter, eg CFL-K0
- Superfluid nuclear matter: suprathermal leptonic viscosity
- Hyperonic nuclear matter
- Investigate effect of multiple equilibrating quantities

Astrophysics:

- Use our results in calculations of r-mode damping times, obtain saturation amplitude of r-mode and trajectory in \((T, \Omega)\) space (requires a cooling model)
- Complications with r-modes: layered stars, role of crust, etc
- Apply to other modes, eg pulsations, f-modes (which emit grav waves)